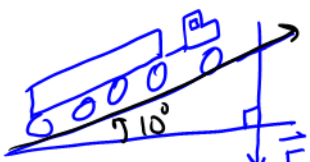
When you are done with your homework you should be able to…

- π Understand the concept of a vector field
- π Determine whether a vector field is conservative
- π Find the curl of a vector field
- π Find the divergence of a vector field

Warm-up: A 48,000-pound truck is parked on a 10° slope. Assume the only force to overcome is that due to gravity.

a. Find the force required to keep the truck from rolling down the hill.



$\vec{F} = -48000\hat{j}$

gravitational force

$\vec{v} = \cos 10^\circ \hat{i} + \sin 10^\circ \hat{j}$

$\vec{w}_1 = \dfrac{\vec{F}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \dfrac{\vec{F}\cdot\vec{v}}{1}\cdot\vec{v} = (-48000)(\sin 10^\circ)\vec{v} \approx -8335.1 \cdot (\cos 10^\circ \hat{i} + \sin 10^\circ \hat{j})$

$\boxed{\|\vec{w}_1\| \approx 8335.1 \text{ lb}}$

b. Find the force perpendicular to the hill.

$\|\vec{w}_2\| = \|\vec{F} - \vec{w}_1\|$

$= \|-48000\hat{j} - (-8335.1\cos 10^\circ \hat{i} - 8335.1 \sin 10^\circ \hat{j})\|$

$= \|8335.1 \cos 10^\circ \hat{i} - 46{,}552.6\hat{j}\|$

$= \boxed{47{,}270.8 \text{ lb}}$

DEFINITION OF VECTOR FIELD

Let M and N be functions of two variables x and y, defined on a plane region R. The function $\mathbf{F}$ defined by $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$ is called a **vector field over *R***. (plane)

Let M, N, and P be functions of three variables x, y and z, defined on a solid region Q. The function $\mathbf{F}$ defined by $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is called a **vector field over *Q***. (space)

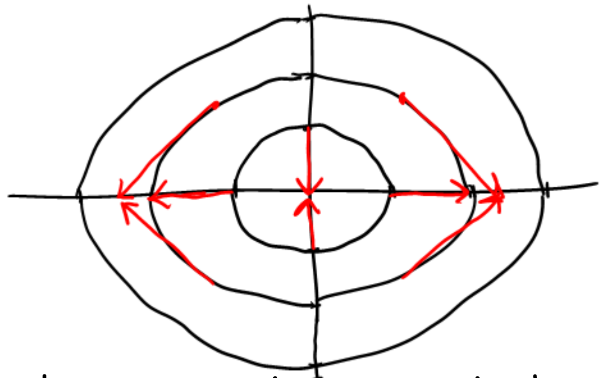
$\vec{F}(1,0) = 1\hat{\imath} - 0\hat{\jmath} = \hat{\imath}$

Example 1: Sketch several representative vectors in the vector field $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$.

$\|\vec{F}\| = \sqrt{x^2+y^2}$

$\|\vec{F}\| = c$

$x^2+y^2 = c^2$

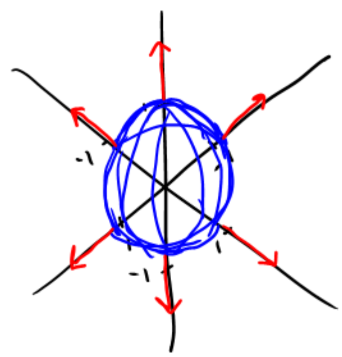


Point	Vector
$(1,0)$	$\hat{\imath}$
$(0,1)$	$-\hat{\jmath}$
$(-1,0)$	$-\hat{\imath}$
$(0,-1)$	$\hat{\jmath}$
$(\sqrt{2},\sqrt{2})$	$\sqrt{2}\hat{\imath}-\sqrt{2}\hat{\jmath}$

Example 2: Sketch several representative vectors in the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$\|\vec{F}\| = c$ and $\|\vec{F}\| = \sqrt{x^2+y^2+z^2}$

$x^2+y^2+z^2 = c^2$



point	vector
$(0,0,1)$	$\hat{k}$
$(0,1,0)$	$\hat{\jmath}$
$(1,0,0)$	$\hat{\imath}$

DEFINITION OF INVERSE SQUARE FIELD

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be a position vector. The vector field $\mathbf{F}$ is an **inverse square field** if

$$\mathbf{F}(x, y, z) = \frac{k}{\|\mathbf{r}\|^2}\mathbf{u}$$

where k is a real number and $\mathbf{u} = \frac{\mathbf{r}}{\|\mathbf{r}\|}$ is a unit vector in the direction of $\mathbf{r}$.

DEFINITION OF CONSERVATIVE VECTOR FIELD

A vector field $\mathbf{F}$ is called **conservative** if there exists a differentiable function f such that $\mathbf{F} = \nabla f$. The function f is called the **potential function** for $\mathbf{F}$.

Example 3: Find the gradient vector field for the scalar function. That is, find the conservative vector field for the potential function.

$$f(x,y,z)=\frac{y}{z}+\frac{z}{x}-\frac{xz}{y}$$

$$\nabla f = f_x\,\hat{\imath} + f_y\,\hat{\jmath} + f_z\,\hat{k} = \vec{F}(x,y,z)$$

$$\boxed{\left(-\frac{z}{x^2}-\frac{z}{y}\right)\hat{\imath} + \left(\frac{1}{z}+\frac{xz}{y^2}\right)\hat{\jmath} + \left(-\frac{y}{z^2}+\frac{1}{x}-\frac{x}{y}\right)\hat{k} = \vec{F}(x,y,z)}$$

THEOREM: TEST FOR CONSERVATIVE VECTOR FIELD IN THE PLANE

Let M and N have continuous first partial derivatives on an open disk R. The vector field given by $\mathbf{F}(x,y)=M\mathbf{i}+N\mathbf{j}$ is conservative if and only if $\frac{dN}{dx}=\frac{dM}{dy}$.

Example 4: Determine whether the vector field is conservative. If it is, find a potential function for the vector field.

$$\mathbf{F}(x,y)=\frac{1}{y^2}(y\mathbf{i}-2x\mathbf{j}) = \frac{1}{y}\hat{\imath} - \frac{2x}{y^2}\hat{\jmath}$$

$M = \frac{1}{y}$ $\qquad$ $N = -\frac{2x}{y^2}$

$\frac{\partial M}{\partial y} = -\frac{1}{y^2}$ $\qquad$ $\frac{\partial N}{\partial x} = -\frac{2}{y^2}$

$-\frac{1}{y^2} \neq -\frac{2}{y^2}$ so $\vec{F}(x,y)$ is not conservative.

DEFINITION OF A CURL OF A VECTOR FIELD

The curl of $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix}$$

$$\begin{aligned}\textbf{curl } \mathbf{F}(x,y,z) &= \nabla \times \mathbf{F}(x,y,z) \\ &= \left(\frac{dP}{dy} - \frac{dN}{dz}\right)\mathbf{i} - \left(\frac{dP}{dx} - \frac{dM}{dz}\right)\mathbf{j} + \left(\frac{dN}{dx} - \frac{dM}{dy}\right)\mathbf{k}\end{aligned}$$

Example 5: Find curl $\mathbf{F}$ for the vector field $\mathbf{F}(x,y,z) = e^{-xyz}(\mathbf{i}+\mathbf{j}+\mathbf{k})$ at the point $(3,2,0)$.

$M = N = P = e^{-xyz}$

$\frac{\partial M}{\partial y} = -xze^{-xyz}$ $\quad \frac{\partial N}{\partial x} = -yze^{-xyz}$ $\quad \frac{\partial P}{\partial x} = -yze^{-xyz}$

$\frac{\partial M}{\partial z} = -xye^{-xyz}$ $\quad \frac{\partial N}{\partial z} = -xye^{-xyz}$ $\quad \frac{\partial P}{\partial y} = -xze^{-xyz}$

$$\boxed{\overrightarrow{\text{curl} F}(x,y,z) = -e^{-xyz}\left[(xz - xy)\hat{i} - (yz - xy)\hat{j} + (yz - xz)\hat{k}\right]}$$

$$\overrightarrow{\text{curl} F}(3,2,0) = -e^{0}\left[(\overset{-6}{3\cdot 0 - 3\cdot 2})\hat{i} - (\overset{6}{2\cdot 0 - 3\cdot 2})\hat{j} + (\overset{0}{2\cdot 0 - 3\cdot 0})\hat{k}\right]$$

$$= \boxed{6\hat{i} - 6\hat{j}}$$

THEOREM: TEST FOR CONSERVATIVE VECTOR FIELD IN SPACE

Suppose that M, N and P have continuous first partial derivatives on an open sphere Q in space. The vector field given by $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative if and only if

$$\textbf{curl } \mathbf{F}(x,y,z) = \mathbf{0}.$$

That is, $\mathbf{F}$ is conservative if and only if

$$\frac{dP}{dy} = \frac{dN}{dz},\ \frac{dP}{dx} = \frac{dM}{dz},\ \text{and}\ \frac{dN}{dx} = \frac{dM}{dy}.$$

Example 6: Determine whether the vector field is conservative. If it is, find a potential function for the vector field.

$M = y^2z^3$, $N = 2xyz^3$, $P = 3xy^2z^2$

$$\mathbf{F}(x, y, z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$$

$\frac{\partial P}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial z}$

$6xyz^2 \stackrel{?}{=} 6xyz^2$ ✓

$\frac{\partial P}{\partial x} \stackrel{?}{=} \frac{\partial M}{\partial z}$

$3y^2z^2 \stackrel{?}{=} 3y^2z^2$ ✓

$\frac{\partial N}{\partial x} \stackrel{?}{=} \frac{\partial M}{\partial y}$

$2yz^3 \stackrel{?}{=} 2yz^3$ ✓

So $\vec{F}$ is conservative

$f_x = y^2z^3$, $f_y = 2xyz^3$, $f_z = 3xy^2z^2$

$f(x,y,z) = \int M\,dx = \int y^2z^3\,dx = xy^2z^3 + g(y,z)$

$f(x,y,z) = \int N\,dy = \int 2xyz^3\,dy = xy^2z^3 + h(x,z)$

$f(x,y,z) = \int P\,dz = \int 3xy^2z^2\,dz = xy^2z^3 + k(x,y)$

So $g(y,z) = h(x,z) = k(x,y) = K$

and $\boxed{f(x,y,z) = xy^2z^3 + K \rightarrow \text{potential function for } \vec{F}(x,y,z)}$

DEFINITION: DIVERGENCE OF A VECTOR FIELD

The **divergence** of $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is

$$\text{div}\,\mathbf{F}(x, y) = \nabla \cdot \mathbf{F}(x, y) = \frac{dM}{dx} + \frac{dN}{dy}.$$

The **divergence** of $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ $\mathbf{F}$ is

$$\text{div}\,\mathbf{F}(x, y, z) = \nabla \cdot \mathbf{F}(x, y, z) = \frac{dM}{dx} + \frac{dN}{dy} + \frac{dP}{dz}.$$

If $\text{div}\,\mathbf{F}(x, y, z) = 0$, then $\mathbf{F}$ is said to be **divergence free**.

Example 7: Find the divergence of the vector field $\mathbf{F}(x, y, z) = \ln(xyz)(\mathbf{i}+\mathbf{j}+\mathbf{k})$ at the point $(3,2,1)$.

$M = \ln(xyz)$ $\quad N = \ln(xyz)$ $\quad P = \ln(xyz)$

$\frac{\partial M}{\partial x} = \frac{yz}{xyz} = \frac{1}{x}$ $\quad \frac{\partial N}{\partial y} = \frac{xz}{xyz} = \frac{1}{y}$ $\quad \frac{\partial P}{\partial z} = \frac{xy}{xyz} = \frac{1}{z}$

$\text{div } \vec{F}(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

$\text{div } \vec{F}(3,2,1) = \frac{1}{3} + \frac{1}{2} + \frac{1}{1} = \boxed{\frac{11}{6}}$

THEOREM: RELATIONSHIP BETWEEN DIVERGENCE AND CURL

If $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field and M, N and P have continuous second partial derivatives, then

$$\text{div}(\textbf{curl } \mathbf{F}) = 0.$$

For vector fields representing velocities of moving particles, the divergence measures the rate of particle flow per unit volume at a point.

In hydrodynamics, the study of fluid motion, a velocity field that is divergence free is called incompressible.

In the study of electricity and magnetism, a vector field that is divergence free is called solenoidal.