4/11/11

- Warm up using 14.7 worksheet
- Lecture 14.7
- Questions

Wednesday

Review

Friday

- Exam 4 / 14.1 - 14.3, 14.5 - 14.7
- HW due

Next Monday

SPRING BREAK!

When you are done with your homework you should be able to…

- π Write and evaluate a triple integral in cylindrical coordinates
- π Write and evaluate a triple integral in spherical coordinates

Warm-up:

1. Find an equation in cylindrical coordinates for the equation $\underbrace{x^2+y^2}_{r^2}-3z^2=0.$

(r, θ, z) is the form of an ordered triple

$r^2-3z^2=0$

$\boxed{r^2=3z^2}$

$0\le\phi\le\pi$

2. Find an equation in spherical coordinates for the equation $x^2+y^2-3z^2=0.$

$(\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2-3(\rho\cos\phi)^2=0$

$\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta-3\rho^2\cos^2\phi=0$

$\rho^2\sin^2\phi(\underbrace{\cos^2\theta+\sin^2\theta}_{1})-3\rho^2\cos^2\phi=0$

$\rho^2(\sin^2\phi-3\cos^2\phi)=0$

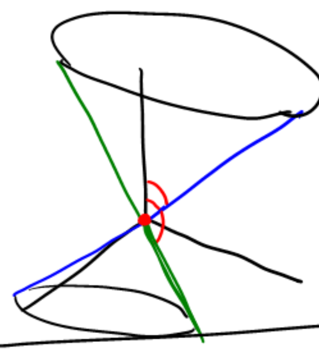
$\rho^2=0$ or $\sin^2\phi=3\cos^2\phi$

$\cancel{\rho=0}$ or $\sqrt{\tan^2\phi}=\sqrt{3}$

included in →

$\tan\phi=\pm\sqrt{3}$

$\phi=\frac{\pi}{3},\frac{2\pi}{3},\cancel{\frac{4\pi}{3}},\cancel{\frac{5\pi}{3}}$



elliptic cone

RECALL: CYLINDRICAL COORDINATES

$$x=r\cos\theta$$
$$y=r\sin\theta$$
$$z=z$$

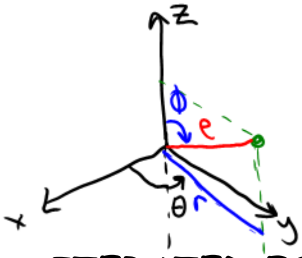
ITERATED FORM OF THE TRIPLE INTEGRAL IN CYLINDRICAL FORM:

If Q is a solid region whose projection R onto the *xy*-plane can be described in polar coordinates, that is, $Q=\{(x,y,z):(x,y) \text{ is in } R,\ h_1(x,y)\le z\le h_2(x,y)\}$ and $R=\{(r,\theta):\theta_1\le\theta\le\theta_2,\ g_1(\theta)\le r\le g_2(\theta)\}$, and if f is a continuous function on the solid Q, you can write the triple integral of f over Q as

$$\iiint_Q f(x,y,z)\,dV=\int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r\cos\theta,r\sin\theta)}^{h_2(r\cos\theta,r\sin\theta)} f(r\cos\theta,r\sin\theta,z)\,r\,dz\,dr\,d\theta$$

RECALL: SPHERICAL COORDINATES

$0 \le \phi \le \pi$



$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

ITERATED FORM OF THE TRIPLE INTEGRAL IN ~~CYLINDRICAL~~ SPHERICAL FORM:

$$\iiint_Q f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

Example 1: Evaluate the iterated integral.

$$-\frac{1}{3}\int_0^{\pi/2}\int_0^{\pi}\int_0^2 -3e^{-\rho^3}\rho^2\, d\rho\, d\theta\, d\phi$$

$$= -\frac{1}{3}\int_0^{\pi/2}\int_0^{\pi} e^{-\rho^3}\Big|_0^2 d\theta\, d\phi$$

$$= -\frac{1}{3}\left(e^{-8}-1\right)\int_0^{\pi/2}\theta\Big|_0^{\pi} d\phi$$

$$= -\frac{\pi}{3}\left(e^{-8}-1\right)\left(\phi\Big|_0^{\pi/2}\right)$$

$$= \boxed{-\frac{\pi^2}{6}\left(e^{-8}-1\right) \text{ or } \frac{\pi^2}{6}\left(1-e^{-8}\right)}$$

Example 2: Sketch the solid region whose volume is given by the iterated integral and evaluate the iterated integral.

$$\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{3-r^2} r\,dz\,dr\,d\theta$$

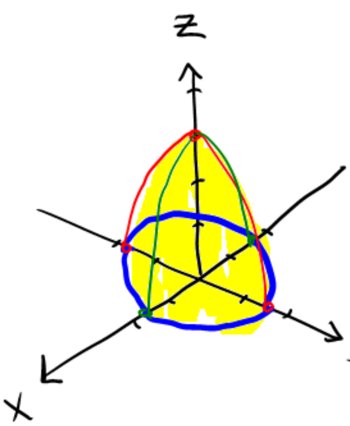
$z=0$ and $z=3-r^2$, $z=3-(x^2+y^2)$

if $z=0$, $x^2+y^2=3$

$x^2+y^2=(\sqrt{3})^2$

$x=0$, $z=3-y^2$

$y=0$, $z=3-x^2$



$$= \int_0^{2\pi}\int_0^{\sqrt{3}} rz\Big|_0^{3-r^2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} r\left[(3-r^2)-0\right]dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} (3r-r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{3r^2}{2}-\frac{r^4}{4}\Big|_0^{\sqrt{3}}\right)d\theta$$

$$= \int_0^{2\pi}\left(\frac{9}{2}-\frac{9}{4}\right)d\theta$$

$$= \frac{9}{4}\int_0^{2\pi} d\theta$$

$$= \frac{9}{4}\left(\theta\Big|_0^{2\pi}\right)$$

$$= \frac{9}{4}\cdot 2\pi$$

$$= \boxed{\frac{9\pi}{2}}$$

Example 3: Convert the integral from rectangular coordinates to both cylindrical and spherical coordinates, and evaluate the simplest iterated integral.

$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{16-x^2-y^2}} \sqrt{x^2+y^2}\,dz\,dy\,dx$$

cylindrical: $\int_0^{\pi/2}\int_0^{2}\int_0^{\sqrt{16-r^2}} r\cdot r\,\partial z\,dr\,\partial\theta = \int_0^{\pi/2}\int_0^{2}\int_0^{\sqrt{16-r^2}} r^2\,\partial z\,dr\,d\theta$

$z=\sqrt{16-x^2-y^2}$, $z=0$

$z=\sqrt{16-(x^2+y^2)}$

$z=\sqrt{16-r^2}$

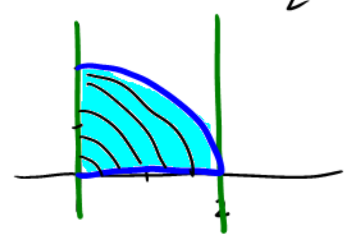
$y=\sqrt{4-x^2}$

$x^2+y^2=4 \to x^2+y^2=(2)^2$

$0\le x\le 2$ plus $y\ge 0$

tells us Quadrant I.

so $0\le\theta\le\frac{\pi}{2}$

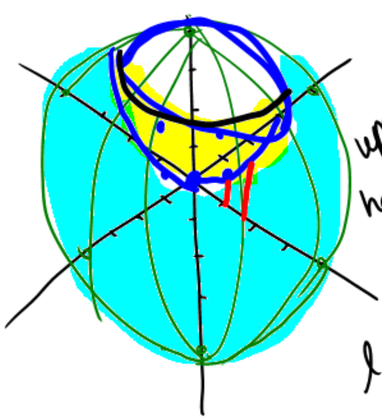


Example 4: Use cylindrical coordinates to find the volume of the solid.

Solid inside $x^2+y^2+z^2=16$ and outside $z=\sqrt{x^2+y^2}$

sphere w/radius of 4

ellipsoid



First octant times 4

upper hemisphere $4\left[\int_0^{\pi/2}\int_0^{2\sqrt{2}}\int_0^{r} r\,dz\,\partial r\,\partial\theta \right.$

$\left. + \int_0^{\pi/2}\int_{2\sqrt{2}}^{4}\int_0^{\sqrt{16-r^2}} r\,dz\,dr\,\partial\theta\right]$

lower hemisphere

$\frac{\frac{4}{3}\pi(4)^3}{2} = \frac{128\pi}{3}$

$z^2=x^2+y^2$

$z^2=r^2$

$z=r$

$z=\pm\sqrt{16-r^2}$

$r=\pm\sqrt{16-r^2}$

$r^2=16-r^2$

$2r^2=16$

$r^2=8$

$r=2\sqrt{2}$

Example 5: Convert the integral from rectangular coordinates to both cylindrical and spherical coordinates, and evaluate the simplest iterated integral.

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$$

$z = \sqrt{9-(x^2+y^2)}$

$z = \sqrt{9-r^2}$

$y = \sqrt{9-x^2}$

$x^2+y^2 = (3)^2$

$0 \le x \le 3$ and $y \ge 0$

quadrant I

cylindrical:

$$\int_0^{\pi/2} \int_0^3 \int_0^{\sqrt{9-r^2}} r\sqrt{r^2+z^2}\, dz\, dr\, d\theta$$

Spherical:

$\rho^2 = x^2+y^2+z^2$

$z = \rho\cos\phi$

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$r^2 = \rho^2\sin^2\phi$

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 \rho \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$z = \sqrt{9-x^2-y^2}$

$z = \sqrt{9-r^2}$

$(z)^2 = \left(\sqrt{9-\rho^2\sin^2\phi}\right)^2$

$\rho^2\cos^2\phi = 9-\rho^2\sin^2\phi$

$\rho^2 \underbrace{(\cos^2\phi+\sin^2\phi)}_{1} = 9$

$\rho^2 = 9$

$\rho = 3, \ \rho \ge 0$