3/14/11

- warm up
- Lecture 13.4

Monday

13.5

When you are done with your homework you should be able to…

- π Understand the concepts of increments and differentials
- π Extend the concept of differentiability to a function of two variables
- π Use a differential as an approximation

Warm-up: The measurement of a side of a square is found to be 12 inches, with a possible error of $\frac{1}{64}$ inch. Use differentials to approximate the possible propagated error in computing the area of the square.

$$\frac{\partial A}{\partial x} = \frac{\partial x^2}{\partial x}$$

$$\frac{\partial A}{\partial x} = 2x$$

$$\partial A = 2x\,\partial x$$

$$\partial A = 2(\cancel{12}^{\,3})\left(\pm\frac{1}{\cancel{64}_{\cancel{32}_{8}}}\right)$$

$$\partial A = \pm\frac{3}{8}\text{ in}^2$$

$\boxed{\text{The possible propagated error in computing the area of the square is } \pm\frac{3}{8}\text{ in}^2.}$

DEFINITION OF TOTAL DIFFERENTIAL

If $z = f(x, y)$ and Δx and Δy are increments of x and y, then the **differentials** of the independent variables x and y are

$$dx = \Delta x \quad \text{and} \quad dy = \Delta y$$

and the **total differential** of the dependent variable z is

$$\partial z = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x, y)\,dx + f_y(x, y)\,dy$$

Example 1: Find the total differential.

a. $z = \frac{x^2}{y}$

$\partial z = f_x(x,y)\,\partial x + f_y(x,y)\,\partial y$

$$\boxed{\partial z = \frac{2x}{y}\,\partial x - \frac{x^2}{y^2}\,\partial y}$$

b. $w = e^y \cos x + z^2$

$\partial w = f_x(x,y,z)\,\partial x + f_y(x,y,z)\,\partial y + f_z(x,y,z)\,\partial z$

$$\boxed{\partial w = -e^y \sin x\,\partial x + e^y \cos x\,\partial y + 2z\,\partial z}$$

DEFINITION OF DIFFERENTIABILITY

A function f given by $z = f(x, y)$ is <u>differentiable</u> at (x_0, y_0) if Δz can be written in the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. The function f is **<u>differentiable</u>** **<u>in a region *R*</u>** if it is differentiable at each point in *R*.

Example 2: Find $z = f(x, y)$ and use the total differential to approximate the quantity.

$$(2.03)^2(1+8.9)^3 - 2^2(1+9)^3$$

$2 + .03$ $9 + (-.1)$ $x = 2,\ y = 9$

$z = x^2(1+y)^3$

$x = 2$

$\partial x = .03$

$y = 9$

$\partial y = -.1$

$\partial z = \frac{\partial z}{\partial x}\partial x + \frac{\partial z}{\partial y}\partial y$

$\partial z = 2x(1+y)^3 \partial x + x^2\left[3(1+y)^2\right]\partial y$

$\partial z = 2(2)(1+9)^3(.03) + (2)^2 3(1+9)^2(-.1)$

$\partial z = 4(30) - 12(10)$

$\partial z = 120 - 120$

$\boxed{\partial z = 0}$

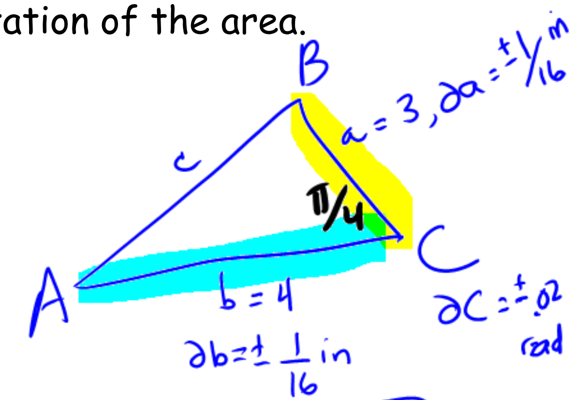
THEOREM: SUFFICIENT CONDITION FOR DIFFERENTIABILITY

If f is a function of x and y, where f_x and f_y are continuous in an open region *R*, then f is differentiable on *R*.

THEOREM: DIFFERENTIABILITY IMPLIES CONTINUITY

If a function of x and y is differentiable at (x_0, y_0) then it is continuous at (x_0, y_0).

Example 3: A triangle is measured and two adjacent sides are found to be 3 inches and 4 inches long, with an included angle of $\frac{\pi}{4}$. The possible errors in measurement are $\frac{1}{16}$ inch for the sides and 0.02 radian for the angle. Approximate the maximum possible error in the computation of the area.



$$Area = \frac{1}{2} a \cdot b \cdot \sin C$$

$$A = \frac{1}{2} ab \sin C$$

$$\partial A = \frac{\partial A}{\partial a}\partial a + \frac{\partial A}{\partial b}\partial b + \frac{\partial A}{\partial C}\partial C$$

$$\partial A = \frac{1}{2}\left[b \sin C \partial a + a \sin C \partial b + ab \cos C \partial C\right]$$

$$\partial A = \frac{1}{2}\left[4\left(\sin \tfrac{\pi}{4}\right)\left(\pm\tfrac{1}{16}\right) + 3\left(\sin \tfrac{\pi}{4}\right)\left(\pm\tfrac{1}{16}\right) + 3 \cdot 4\left(\cos \tfrac{\pi}{4}\right)(\pm.02)\right] \approx \boxed{\pm 0.239}$$

Example 4: Show that the function $f(x,y) = x^2 + y^2$ is ~~differentiable~~ continuous by finding values for ε_1 and ε_2 as designated in the definition of differentiability, and verify that both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

DEFINITION OF DIFFERENTIABILITY

A function f given by $z = f(x,y)$ is <u>differentiable</u> at (x_0, y_0) if Δz can be written in the form

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. The function f is **differentiable in a region *R*** if it is differentiable at each point in *R*.

$$\Delta z = \left[f_x(x_0,y_0) + \varepsilon_1\right]\Delta x + \left[f_y(x_0,y_0) + \varepsilon_2\right]\Delta y \text{ where}$$

ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

$$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$$

let $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$

$$f(x,y) - f(x_0,y_0) = [f_x(x_0,y_0)+\varepsilon_1]\Delta x + [f_y(x_0,y_0)+\varepsilon_2]\Delta y$$

$$= [f_x(x_0,y_0)+\varepsilon_1](x-x_0) + [f_y(x_0,y_0)+\varepsilon_2](y-y_0)$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = [f_x(x_0,y_0)+\varepsilon_1](x-x_0) + [f_y(x_0,y_0)+\varepsilon_2](y-y_0) + f(x_0,y_0)$$

$$= 0 + f(x_0,y_0)$$

$$= f(x_0,y_0)$$

So f is continuous at (x_0,y_0).

Example 4: Show that the function $f(x,y) = x^2+y^2$ is differentiable by finding values for ε_1 and ε_2 as designated in the definition of differentiability, and verify that both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Let $z = f(x,y)$

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= [(x+\Delta x)^2 + (y+\Delta y)^2] - (x^2+y^2)$$

$$= \cancel{x^2} + 2x\Delta x + (\Delta x)^2 + \cancel{y^2} + 2y\Delta y + (\Delta y)^2 - \cancel{x^2} - \cancel{y^2}$$

$$= 2x(\Delta x) + 2y(\Delta y) + \Delta x(\Delta x) + \Delta y(\Delta y)$$

$$= f_x(x,y)(\Delta x) + f_y(x,y)(\Delta y) + \varepsilon_1(\Delta x) + \varepsilon_2(\Delta y)$$

$\varepsilon_1 = \Delta x$, $\varepsilon_2 = \Delta y$ as $(\Delta x, \Delta y) \to (0,0)$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ //