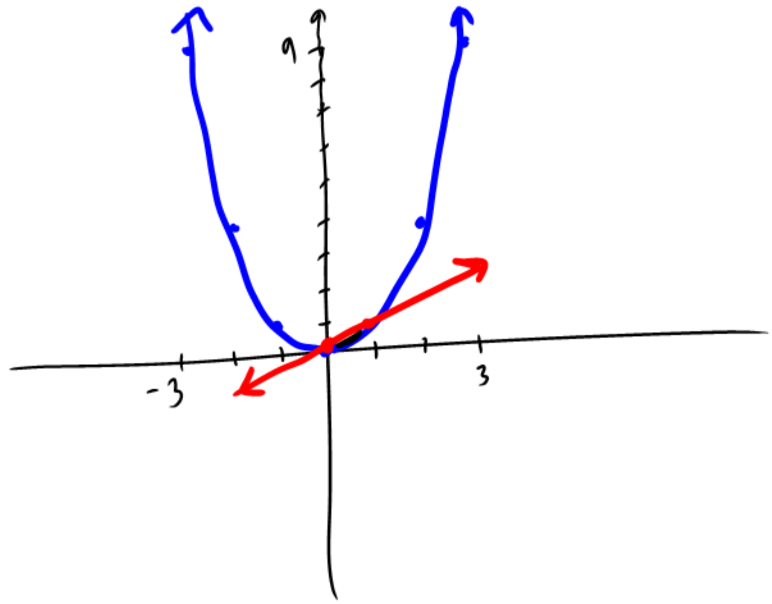
3/30/11

- Make sure you're good with 14.1
- Lecture 14.2

Friday

- Finish 14.2
- Lecture 14.3

Find the area bounded by the graphs of $y = x^2$ and $y = x$.



$x = x^2$

$0 = x^2 - x$

$0 = x(x-1)$

$x = 0,\ x = 1$

$$A = \int_0^1 (x - x^2)\,dx$$

$$A = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1$$

Consider...

$$\int_{x^2}^{x} dy = y\Big|_{x^2}^{x}$$

$$= x - x^2$$

$$A = \int_0^1 \left[\int_{x^2}^{x} dy\right] dx$$

equivalent

When you are done with your homework you should be able to…

- π Evaluate an iterated integral
- π Use an iterated integral to find the area of a plane region

Warm-up: Sketch the region bounded by the graphs $x = \cos y,\ x = \frac{1}{2},\ \frac{\pi}{3} \le y \le \frac{7\pi}{3}$.

Then find the area.

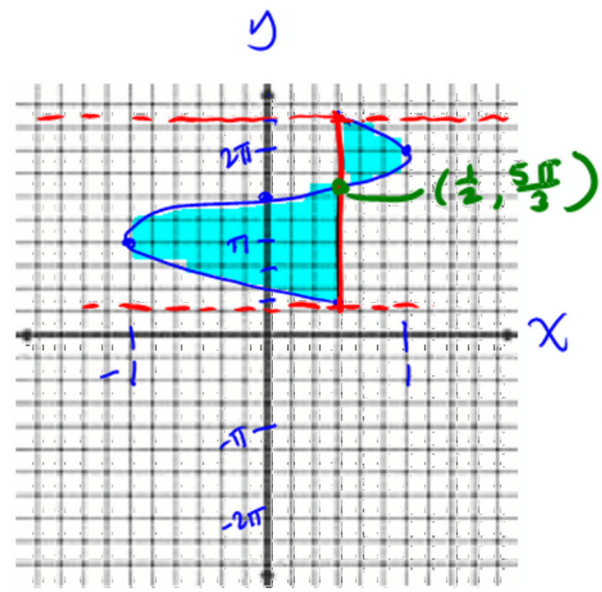
$$A = \int_{\pi/3}^{5\pi/3} \left(\tfrac{1}{2} - \cos y\right) dy + \int_{5\pi/3}^{7\pi/3} \left(\cos y - \tfrac{1}{2}\right) dy$$

$$A = \left(\frac{y}{2} - \sin y\right)\Big|_{\pi/3}^{5\pi/3} + \left(\sin y - \frac{y}{2}\right)\Big|_{5\pi/3}^{7\pi/3}$$

$$A = \left[\left(\frac{5\pi/3}{2} - \sin 5\pi/3\right) - \left(\frac{\pi/3}{2} - \sin \pi/3\right)\right] + \left[\left(\sin 7\pi/3 - \frac{7\pi/3}{2}\right) - \left(\sin 5\pi/3 - \frac{5\pi/3}{2}\right)\right]$$

$$A = \frac{5\pi}{6} + \frac{\sqrt{3}}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} - \frac{7\pi}{6} + \frac{\sqrt{3}}{2} + \frac{5\pi}{6}$$

$$\boxed{A = \left(\pi/3 + 2\sqrt{3}\right) \text{sq. units}}$$



INTEGRALS OF FUNCTIONS OF TWO VARIABLES

1. When integrating a function of two variables with respect to x, you hold y constant: $\int_{h_1(y)}^{h_2(y)} f_x(x,y)\,dx = f(x,y)\Big]_{h_1(y)}^{h_2(y)} = f(h_2(y),y) - f(h_1(y),y)$.
2. When integrating a function of two variables with respect to y, you hold x constant: $\int_{g_1(x)}^{g_2(x)} f_y(x,y)\,dy = f(x,y)\Big]_{g_1(x)}^{g_2(x)} = f(x,g_2(x)) - f(x,g_1(x))$.

Example 1: Evaluate the following integrals.

a. $\int_x^{x^2} \frac{y}{x} dy = \frac{1}{x}\int_x^{x^2} y\,dy$

$= \frac{1}{x}\left(\frac{y^2}{2}\right)\Big|_x^{x^2}$

$= \frac{1}{2x}\left((x^2)^2 - (x)^2\right)$

$= \frac{1}{2x}(x^4 - x^2)$

$= \frac{1}{2}(x^3 - x)$

$= \boxed{\frac{x}{2}(x^2-1)}$

b. $\int_y^{\pi/2} \sin^3 x \cos y\,dx = \cos y \int_y^{\pi/2} \sin^2 x \sin x\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$= \cos y \int_y^{\pi/2} (1-\cos^2 x)\sin x\,dx$

$= \cos y \int_y^{\pi/2} \left(\sin x + (\cos x)^2(-\sin x)\right)dx$

$= \cos y\left[-\cos x + \frac{\cos^3 x}{3}\right]_y^{\pi/2}$

$= \cos y\left[\left(-\cos \pi/2 + \frac{(\cos \pi/2)^3}{3}\right) - \left(-\cos y + \frac{\cos^3 y}{3}\right)\right]$

$= \cos y\left(\cos y - \frac{\cos^3 y}{3}\right)$

$= \boxed{\frac{\cos^2 y}{3}(3-\cos^2 y)}$

ITERATED INTEGRALS

When evaluating the integral of an integral, it is called an ***iterated integral***.

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx = \int_a^b f(x,y)\Big]_{g_1(x)}^{g_2(x)} dx$$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy = \int_c^d f(x,y)\Big]_{h_1(y)}^{h_2(y)} dy$$

Example 2: Evaluate the following iterated integrals.

a. $\int_0^1\int_0^2 (x+y)\,dy\,dx$ $= \int_0^1 \left(yx + y^2/2\right|_0^2 dx$

$= \int_0^1 \left[(2x + 4/2) - (0+0)\right] dx$

$= \int_0^1 (2x+2)\,dx$

$= \left(x^2 + 2x\right|_0^1$

$= \left[(1+2) - (0+0)\right]$

$= \boxed{3}$

b. $\int_1^4\int_1^{\sqrt{x}} 2ye^{-x}\,dy\,dx$ $= \int_1^4 e^{-x} \left(y^2\right|_1^{\sqrt{x}} dx$

$u = x-1$

$du = dx$

$dv = e^{-x}dx$

$v = -e^{-x}$

$= \int_1^4 e^{-x}\left[(\sqrt{x})^2 - (1)^2\right] dx$

$= \int_1^4 e^{-x}(x-1)\,dx$

$= -e^{-x}(x-1) + \int e^{-x}\,dx$

$= \left(-e^{-x}(x-1) - e^{-x}\right|_1^4$

$= -e^{-x}\left[(x-1)+1\right]_1^4$

$= -xe^{-x}\Big|_1^4$

$= \boxed{-4e^{-4} + e^{-1}}$

c. $\int_0^3\int_0^\infty \frac{x^2}{1+y^2}\,dy\,dx$

$\int_0^\infty \frac{1}{1+y^2}\,dy = \lim_{b\to\infty} \int_0^b \frac{1}{1+y^2}\,dy$

$= \lim_{b\to\infty} \arctan y \Big|_0^b$

$= \lim_{b\to\infty} \arctan b - \lim_{b\to\infty} \arctan 0$

$= \frac{\pi}{2} - \lim_{b\to\infty} 0$

$= \frac{\pi}{2} - 0$

$= \frac{\pi}{2}$

So... $\int_0^3\int_0^\infty \frac{x^2}{1+y^2}\,dy\,dx$

$= \int_0^3 \frac{\pi}{2}x^2\,dx$

$= \frac{\pi}{6}x^3\Big|_0^3$

$= \frac{\pi}{6}(27-0)$

$= \frac{27\pi}{6}$

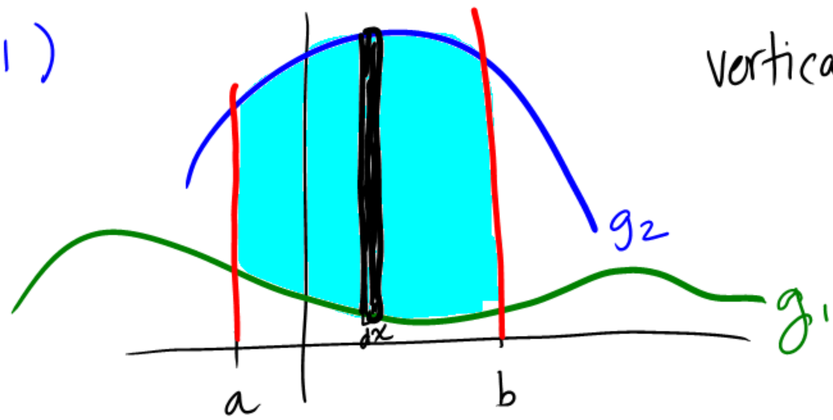
$= \boxed{\frac{9\pi}{2}}$

AREA OF A REGION IN THE PLANE

If R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on $[a,b]$, then the area of R is given by

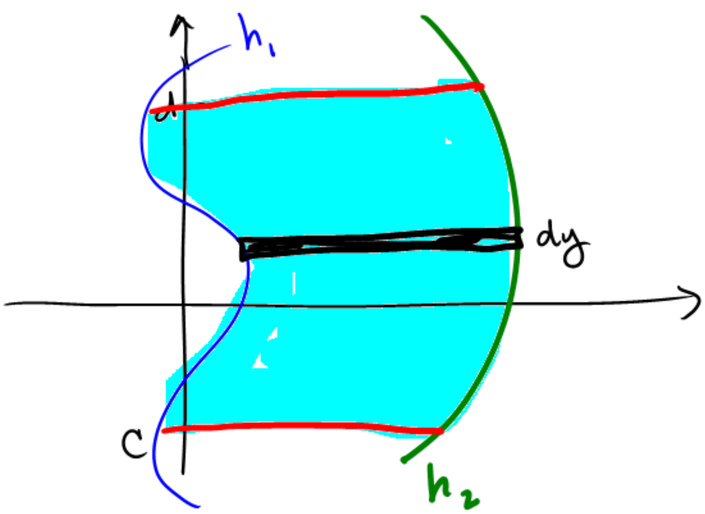
1. $A = \int_a^b \int_{g_1(x)}^{g_2(x)} dydx$ (vertically simple)
2. $A = \int_c^d \int_{h_1(y)}^{h_2(y)} dxdy$ (horizontally simple)

(1)



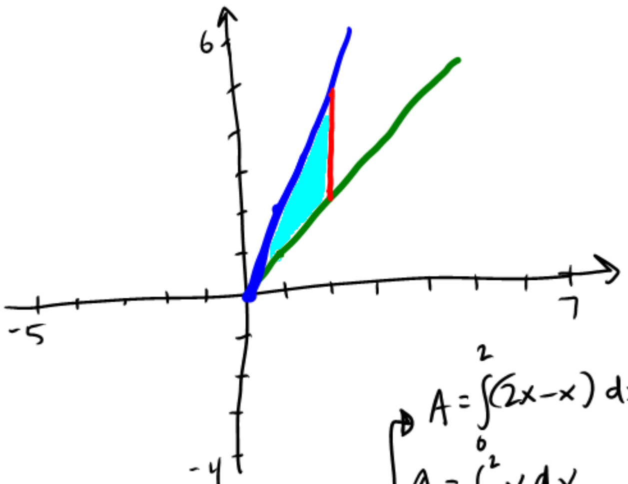
vertically simple

(2)

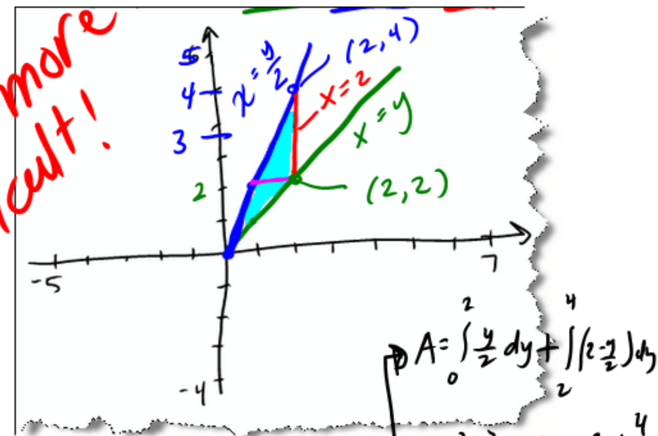


horizontally simple

Example 3: Use an iterated integral to find the area of the region bounded by the graphs of $y = x$, $y = 2x$, $x = 2$.



Much more difficult!



Vertically simple:

$$A = \int_0^2 \int_x^{2x} dy\,dx$$

$$A = \int_0^2 y\Big|_x^{2x} dx$$

$$A = \int_0^2 (2x - x)\,dx$$

$$A = \int_0^2 x\,dx$$

$$A = \frac{x^2}{2}\Big|_0^2$$

$$\boxed{A = 2 \text{ sq. units}}$$

Horizontally Simple:

$$A = \int_0^2 \int_{\frac{y}{2}}^{y} dx\,dy + \int_2^4 \int_{y/2}^{2} dx\,dy$$

$$A = \int_0^2 (y - y/2)\,dy + \int_2^4 (2 - y/2)\,dy$$

$$A = \int_0^2 \frac{y}{2}\,dy + \int_2^4 \left(2 - \frac{y}{2}\right)dy$$

$$A = \frac{y^2}{4}\Big|_0^2 + \left(2y - \frac{y^2}{4}\right)\Big|_2^4$$

$$A = (1 - 0) + [(8-4) - (4-1)]$$

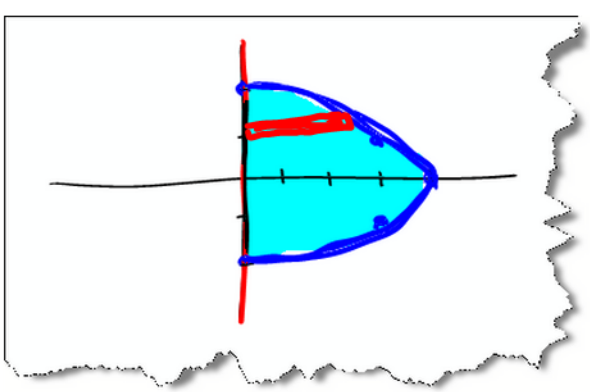
$$A = 1 + 4 - 3$$

$$\boxed{A = 2 \text{ sq. units}}$$

Example 4: Sketch the region R whose area is given by the iterated integral. Then switch the order of integration and show that both orders yield the same area.

$$\int_{-2}^{2} \int_0^{4-y^2} dx\,dy$$

$x = 0$, $x = 4 - y^2$



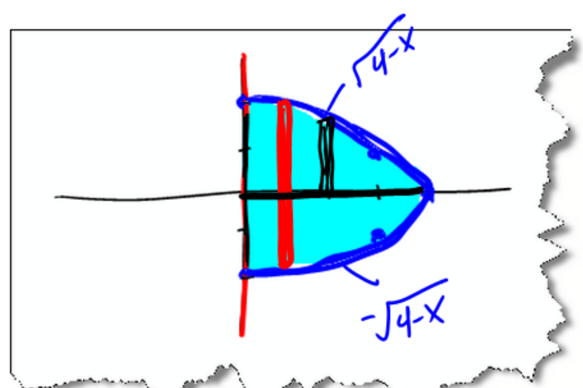
horizontally simple

2 options: ① cut in half and double the area

② consider both halves

$x = 4 - y^2$

$y = \pm\sqrt{4-x}$



want vertically simple

① $2\int_0^4 \int_0^{\sqrt{4-x}} dy\,dx$

② $\int_0^4 \int_{-\sqrt{4-x}}^{\sqrt{4-x}} dy\,dx$