3/2/11

- Lecture 13.3

* 13.2 homework is extra credit

Friday

13.3

13.2

(12) $\lim_{(x,y)\to(2,4)} \left(\frac{x+y}{x^2+1}\right) = \frac{2+4}{2^2+1}$

$= \boxed{\frac{6}{5}}$

When you are done with your homework you should be able to…

- π Find and use partial derivatives of a function of two variables
- π Find and use partial derivatives of a function of three or more variables
- π Find higher-order partial derivatives of a function of two or three variables

Warm-up: Find the derivative of the following functions. Simplify your result to a single rational expression with positive exponents.

1. $f(x)=\dfrac{3x^2-x+2}{\sqrt{x}} = 3x^{3/2}-x^{1/2}+2x^{-1/2}$

$f'(x)=\frac{9}{2}x^{1/2}-\frac{1}{2}x^{-1/2}-x^{-3/2}$

$\boxed{f'(x)=\frac{1}{2x^{3/2}}\left(9x^2-x-2\right)}$

2. $g(x)=(5x-3)^2$

$g'(x)=2(5x-3)^1\cdot 5$

$\boxed{g'(x)=10(5x-3)}$

3. $f(x)=\cos\left(x-\frac{\pi}{4}\right)$

$f'(x)=-\sin\left(x-\frac{\pi}{4}\right)\cdot 1$

$\boxed{f'(x)=-\sin\left(x-\frac{\pi}{4}\right)}$

DEFINITION: PARTIAL DERIVATIVES OF A FUNCTION OF TWO VARIABLES

If $z = f(x, y)$ then the **first partial derivatives** of f with respect to x and y are f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided the limit exists.

Example 1: Find the partial derivatives f_x and f_y of the following functions.

a. $f(x, y) = x^2 - 2y^2 + 4$

$f_x(x,y) = 2x - 0 + 0$

$\boxed{f_x(x,y) = 2x}$

$f_y(x,y) = 0 - 4y + 0$

$\boxed{f_y(x,y) = -4y}$

b. $z = \sin 5x \cos 5y$

$z_x = \cos 5y [5\cos 5x]$

$\boxed{z_x = 5\cos 5x \cos 5y}$

$z_y = \sin 5x [-5\sin 5y]$

$\boxed{z_y = -5\sin 5x \sin 5y}$

c. $f(x, y) = \int_x^y (2t+1)\,dt + \int_y^x (2t-1)\,dt$

$f(x,y) = \int_x^y (2t+1)\,dt - \int_x^y (2t-1)\,dt$

$f(x,y) = \int_x^y 2\,dt$

$f(x,y) = 2(t \Big|_x^y$

$f(x,y) = 2(y-x)$

$\boxed{f_x(x,y) = -2}$

$\boxed{f_y(x,y) = 2}$

NOTATION FOR FIRST PARTIAL DERIVATIVES FOR $z=f(x,y)$

$$\frac{\partial}{\partial x} f(x,y) = f_x(x,y) = z_x = \frac{\partial z}{\partial x}$$

and

$$\frac{\partial}{\partial y} f(x,y) = f_y(x,y) = z_y = \frac{\partial z}{\partial y}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = f_x(a,b)$$

and

$$\left.\frac{\partial z}{\partial y}\right|_{(a,b)} = f_y(a,b)$$

Example 2: Use the limit definition to find the first partial derivatives with respect to x, y and z.

$$f(x,y,z) = 3x^2y - 5xyz + 10yz^2$$

$$f_x(x,y,z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[3(x+\Delta x)^2 y - 5(x+\Delta x)yz + \cancel{10yz^2}\right] - \left(3x^2y - 5xyz + \cancel{10yz^2}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{3x^2y} + 6x\cancel{\Delta x}y + 3(\Delta x)^{\cancel{2}}y \cancel{- 5xyz} - 5\cancel{\Delta x}yz \cancel{- 3x^2y} \cancel{+ 5xyz}}{\cancel{\Delta x}}$$

$$= 6xy + 3(0)y - 5yz$$

$$= 6xy - 5yz$$

$$f_y(x,y,z) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y, z) - f(x,y,z)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\left[3x^2(y+\Delta y) - 5x(y+\Delta y)z + 10(y+\Delta y)z^2\right] - \left(3x^2y - 5xyz + 10yz^2\right)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\cancel{3x^2y} + 3x^2\cancel{\Delta y} - \cancel{5xyz} - 5x\cancel{\Delta y}z + \cancel{10yz^2} + 10\cancel{\Delta y}z^2 - \cancel{3x^2y} + \cancel{5xyz} - \cancel{10yz^2}}{\cancel{\Delta y}}$$

$$= \boxed{3x^2 - 5xz + 10z^2}$$

PARTIAL DERIVATIVES OF A FUNCTION OF THREE OR MORE VARIABLES

If $w = f(x, y, z)$ then the **first partial derivatives** of f with respect to x, y and z are defined by

$$\frac{dw}{dx} = f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x}$$
$$\frac{dw}{dy} = f_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y, z) - f(x, y, z)}{\Delta y}$$
$$\frac{dw}{dz} = f_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z+\Delta z) - f(x, y, z)}{\Delta z}$$

provided the limit exists.

Example 3: Find f_x, f_y and f_z at the given point.

$$f(x, y, z) = \frac{xy}{x+y+z}, \quad (3, 1, -1)$$

$$f_x(x,y,z) = \frac{y(x+y+z) - xy(1)}{(x+y+z)^2}\Bigg|_{(3,1,-1)} = \frac{(1)(3+1+(-1)) - (3)(1)}{(3+1+(-1))^2}$$

$$= \boxed{0}$$

$$f_y(x,y,z) = \frac{x(x+y+z) - xy(1)}{(x+y+z)^2}\Bigg|_{(3,1,-1)} = \frac{3(3+1+(-1)) - (3)(1)}{(3+1+(-1))^2}$$

$$= \frac{9-3}{3^2}$$

$$= \boxed{\frac{2}{3}}$$

$$f(x,y,z) = xy(x+y+z)^{-1}$$

$$f_z(x,y,z) = xy\left[-(x+y+z)^{-2}\cdot 1\right]\Big|_{(3,1,-1)} = 3(1)\left[-(3+1+(-1))^{-2}\right]$$

$$= \boxed{-\frac{1}{3}}$$

HIGHER ORDER PARTIAL DERIVATIVES

1. Differentiate twice with respect to *x*.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=f_{xx}$$

2. Differentiate twice with respect to *y*.

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=f_{yy}$$

3. Differentiate first with respect to *x* and then with respect to *y*.

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x}=f_{xy}$$

4. Differentiate first with respect to *y* and then with respect to *x*.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\partial y}=f_{yx}$$

Example 4: Find the four second partial derivatives.

a. $z=\ln(x-y)$

$z_x=\frac{1}{x-y}$

$z_y=-\frac{1}{x-y}$

$$\frac{\partial}{\partial x}(z_x)=z_{xx}=\boxed{-\frac{1}{(x-y)^2}}$$

$$\frac{\partial}{\partial y}(z_x)=z_{xy}=-\frac{-1}{(x-y)^2}=\boxed{\frac{1}{(x-y)^2}}$$

always the same

$$z_{yx}=-1\left(-\frac{1}{(x-y)^2}\right)=\boxed{\frac{1}{(x-y)^2}}$$

$$z_{yy}=\frac{\partial}{\partial y}\left[-(x-y)^{-1}\right]$$

$$=(x-y)^{-2}(-1)$$

$$=\boxed{-\frac{1}{(x-y)^2}}$$

b. $z=\arctan\left(\frac{y}{x}\right)$

THEOREM: EQUALITY OF MIXED PARTIAL DERIVATIVES

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then, for every (x, y) in R,

$$f_{xy}(x, y) = f_{yx}(x, y)$$

Example 5: Find the slopes of the surface in the *x*- and *y*-directions at the given point.

$h(x, y) = x^2 - y^2, \quad (-2,1,3)$

$$h_x(x,y) = 2x \Big|_{(-2,1,3)} = 2(-2) = -4$$

$$h_y(x,y) = -2y \Big|_{(-2,1,3)} = -2(1) = -2$$

The slope of the surface in the x-direction is −4 and the slope of the surface in the y-direction is −2.