3/14/11

·warm up ·Lecture 13.8	Wednesday Lecture 13.9	Friday Review	Next Monday Exam 3/Ch.13 HW Due (13.2 extra credit)

3.8

(12) Examine the function for relative extrema

$f(x,y) = -5x^2 + 4xy - y^2 + 16x + 10$

$f_x(x,y) = -10x + 4y + 16$

$f_y(x,y) = 4x - 2y$

critical point

$-10x + 4y + 16 = 0$
$4x - 2y = 0$

$-5x + 2y = -8$
$4x - 2y = 0$

$-x = -8$

$x = 8$

$4x - 2y = 0$

$4(8) - 2y = 0$

$-2y = -32$

$y = 16$

critical point is $(8,16)$

$f_{xx}(x,y) = -10$

so $f_{xx}(8,16) = -10$

$f_{yy}(x,y) = -2$

$f_{yy}(8,16) = -2$

$f_{xy}(x,y) = 4$

$f_{xy}(8,16) = 4$

$d = (-10)(-2) - (4)^2$

$d = 20 - 16$

$d = 4$

$f_{xx}(8,16) = -10 < 0$ and $d > 0$

So there is a relative maximum at $(8, 16, 74)$

$f(8,16) = -5(8)^2 + 4(8)(16) - (16)^2 + 16(8) + 10$

$= 74$

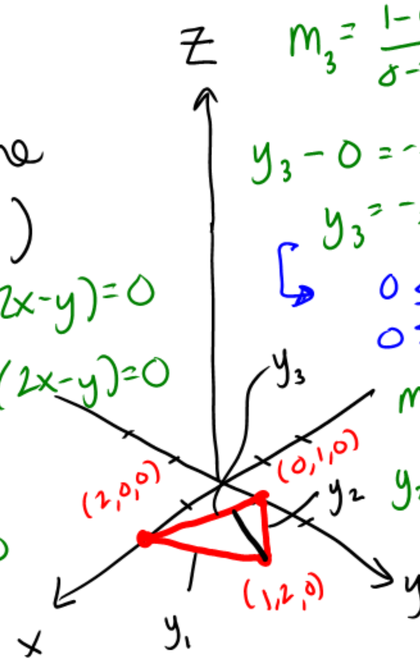
(48) $f(x,y) = (2x-y)^2$

R: The triangular region in the xy-plane with vertices $(2,0)$, $(0,1)$ and $(1,2)$

$f_x(x,y) = 2(2x-y)^1 \cdot 2 = 4(2x-y)$

$f_y(x,y) = 2(2x-y)^1 \cdot (-1) = -2(2x-y)$

$\left\{\begin{array}{l} 4(2x-y)=0 \\ -2(2x-y)=0 \end{array}\right. \rightarrow 2x-y=0$



$m_3 = \frac{1-0}{0-2} = -\frac{1}{2}$

$y_3 - 0 = -\frac{1}{2}(x-2)$

$y_3 = -\frac{1}{2}x + 1$

$\rightarrow 0 \le x \le 2$, $0 \le y_3 \le 1$

$m_2 = \frac{2-1}{1-0} = 1$

$y_2 - 1 = 1(x-0)$

$y_2 = x+1$

$\rightarrow 0 \le x \le 1$, $1 \le y_2 \le 2$

$m_1 = \frac{2-0}{1-2} = -2$

$y_1 - 0 = -2(x-2)$

$y_1 = -2x+4$

$\rightarrow 1 \le x \le 2$, $0 \le y_1 \le 2$

On the line y_1:

$2x - (-2x+4) = 0$

$2x + 2x - 4 = 0$

$4x = 4$

$x = 1, y = 2$

crit. point: $(1,2)$

On the line y_2:

$2x - (x+1) = 0$

$x - 1 = 0$

$x = 1$

$y_2 = 2$

crit. point: $(1,2)$

On the line y_3:

$2x - (-\frac{1}{2}x + 1) = 0$

$2x + \frac{1}{2}x - 1 = 0$

$\frac{5}{2}x = 1$

$x = \frac{2}{5}$

$y_3 = -\frac{1}{2}(\frac{2}{5}) + 1 = \frac{4}{5}$

crit point $(\frac{2}{5}, \frac{4}{5})$

$f_{xx}(x,y) = 8$

$f_{yy}(x,y) = 2$

$f_{xy}(x,y) = -4$

$d = (8)(2) - (-4)^2$

$d = 0$ So we need to examine the function values

crit. values $\left\{\begin{array}{l} f(1,2) = [(2(1) - (2)]^2 = 0 \\ f(\frac{2}{5}, \frac{4}{5}) = [2(\frac{2}{5}) - \frac{4}{5}]^2 = 0 \end{array}\right.$

$z = (2x-y)^2$

$0 = 2x - y$

$y = 2x$

vertices

$f(2,0) = [2(2)-0]^2 = 16$

$f(1,2) = 0$ (already found)

$f(0,1) = [2(0)-1]^2 = 1$

Absolute max is at $(2,0,16)$ and absolute min is 0 at $(1,2,0)$ and along the line $y=2x$

When you are done with your homework you should be able to…

- π Find the absolute and relative extrema of a function of two variables
- π Use the Second Partials Test to find relative extrema of a function of two variables

$f(x) = \frac{\sin 2x}{2}$

Warm-up: Consider the function $f(x) = \sin x \cos x$ on the interval $(0, \pi)$.

A. Find the critical numbers.

$0 < x < \pi$

$0 < 2x < 2\pi$

$f'(x) = \cancel{\frac{1}{2}}\left[\cancel{2}\cos 2x\right]$

$0 = \cos 2x$

$2x = \frac{\pi}{2}, \frac{3\pi}{2} \rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$

$\boxed{\text{critical \#'s: } \left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}}$

B. Apply the theorem which tests for increasing and decreasing intervals.

$f'(x) = \cos 2x$

$f'\left(\frac{\pi}{6}\right) = \frac{1}{2} > 0$

$f'\left(\frac{\pi}{2}\right) = -1 < 0$

$f'\left(\frac{5\pi}{6}\right) = \frac{1}{2} > 0$

$\left(0, \frac{\pi}{4}\right)$	$\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$	$\left(\frac{3\pi}{4}, \pi\right)$
+	−	+

$0 \quad \frac{\pi}{6} \quad \frac{\pi}{4} \quad \frac{\pi}{2} \quad \frac{3\pi}{4} \quad \frac{5\pi}{6} \quad \pi$

C. Find the open interval(s) on which the function is

a. Increasing

b. Decreasing

f is increasing on $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right)$

f is decreasing on $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

D. Apply the First Derivative test to identify all relative extrema. Give your result(s) as an ordered pair.

At $\left(\frac{\pi}{4}, f\left(\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{4}, \frac{1}{2}\right)$ there is a relative maximum and at $\left(\frac{3\pi}{4}, -\frac{1}{2}\right)$ there's a relative minimum.

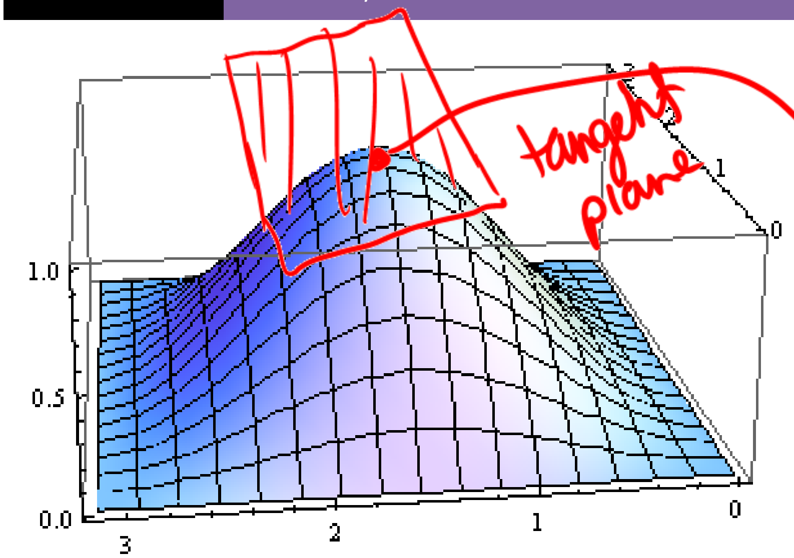
$f(x) = \frac{1}{2}\sin 2x$

$f\left(\frac{\pi}{4}\right) = \frac{1}{2}\sin 2\left(\frac{\pi}{4}\right)$

$= \frac{1}{2}\sin\frac{\pi}{2}$

$= \frac{1}{2}\cdot 1$

$= \frac{1}{2}$



ordered triple where f takes on a maximum value

Plot3D[Sin[x]Sin[y]^2, {x, 0, Pi}, {y, 0, Pi}]

THEOREM: EXTREME VALUE THEOREM

Let f be a continuous function of two variables x and y defined on a closed bounded region R in the *xy*-plane.

1. There is at least one point in R where f takes on a minimum value.
2. There is at least one point in R where f takes on a maximum value.

DEFINITION: RELATIVE EXTREMA

Let f be a function defined on a region R containing (x_0, y_0).

1. The function f has a **relative minimum** at (x_0, y_0) if $f(x,y) \geq f(x_0, y_0)$ for all x and y in an *open* disk containing (x_0, y_0).
2. The function f has a **relative maximum** at (x_0, y_0) if $f(x,y) \leq f(x_0, y_0)$ for all x and y in an *open* disk containing (x_0, y_0).

DEFINITION: CRITICAL POINT

Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a **critical point** of f if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist

THEOREM: RELATIVE EXTREMA OCCUR ONLY AT CRITICAL POINTS

If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f.

THEOREM: SECOND PARTIALS TEST

Let f have continuous partial derivatives on an open region containing a point (a,b) for which $f_x(a,b)=0$ and $f_y(a,b)=0$. To test for relative extrema of f, consider the quantity $d = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$.

1. If $d > 0$ and $f_{xx}(a,b) > 0$, then f has a **relative minimum** at (a,b).
2. If $d > 0$ and $f_{xx}(a,b) < 0$, then f has a **relative maximum** at (a,b).
3. If $d < 0$, then $(a,b,f(a,b))$ is a **saddle point**.
4. The test is inconclusive if If $d = 0$.

Example 1: Examine the function for relative extrema and saddle points.

$$g(x,y) = xy$$

$g_x(x,y) = y \rightarrow 0 = y \rightarrow y = 0 = b$

$g_{xx}(x,y) = 0 \rightarrow g_{xx}(0,0) = 0$

$g_y(x,y) = x \rightarrow 0 = x \rightarrow x = 0 = a$

$g_{yy}(x,y) = 0 \rightarrow g_{yy}(0,0) = 0$

$g_{xy}(x,y) = 1 \rightarrow g_{xy}(0,0) = 1$

$d = g_{xx}(0,0) \cdot g_{yy}(0,0) - \left[g_{xy}(0,0)\right]^2$

$d = 0 \cdot 0 - (1)^2$

$d = -1$

$(0,0)$ is a critical point

$d = -1 < 0$

So there is a saddle point at $(0,0,g(0,0)) = (0,0,0)$

Example 2: Find the critical points and test for relative extrema. List the critical points for which the Second Partials Test fails.

$$f(x,y) = x^3 + y^3 - 6x^2 + 9y^2 + 12x + 27y + 19$$

$f_x(x,y) = 3x^2 - 12x + 12$

$f_{xx}(x,y) = 6x - 12$

$f_{xx}(2,-3) = 6\cdot 2 - 12 = 0$

$f_y(x,y) = 3y^2 + 18y + 27$

$f_{yy}(x,y) = 6y + 18$

$f_{yy}(2,-3) = 6(-3) + 18 = 0$

$f_{xy}(x,y) = 0$

critical point

$3x^2 - 12x + 12 = 0$ and $3y^2 + 18y + 27 = 0$

$x^2 - 4x + 4 = 0$ $\quad$ $y^2 + 6y + 9 = 0$

$(x-2)^2 = 0$ $\quad$ $(y+3)^2 = 0$

$x - 2 = 0$ $\quad$ $y + 3 = 0$

$x = 2$ $\quad$ $y = -3$

critical point is $(2,-3)$

$d = 0\cdot 0 - 0^2 \rightarrow d = 0$ $\boxed{\text{so the 2nd partials test fails at } (2,-3)}$

Example 3: A function f has continuous second partial derivatives on an open region containing the critical point (a,b). If $f_{xx}(a,b)$ and $f_{yy}(a,b)$ have opposite signs, what is implied?

$$d = f_{xx}(a,b)\,f_{yy}(a,b) - \underbrace{[f_{xy}(a,b)]^2}_{\text{positive}}$$

$= (-)\cdot(+) - \text{positive}$

$=$ negative subtract a positive

$=$ negative

$\Rightarrow$ saddle point