2/11/11

- Warm up
- Finish 12.3
- Lecture 12.4

Wednesday

12.5

Wed. 2/23

Review

Next exam is 2/25/11

When you are done with your homework you should be able to…

- π Find a unit tangent vector at a point on a space curve
- π Find the tangential and normal components of acceleration

Warm-up: Consider the two curves given by $y_1 = 1 - x^2$ and $y_2 = x^2 - 1$.

a. Find the unit tangent vectors to each curve at their points of intersection.

Point of intersection

$1 - x^2 = x^2 - 1$

$2 = 2x^2$

$x^2 = 1$

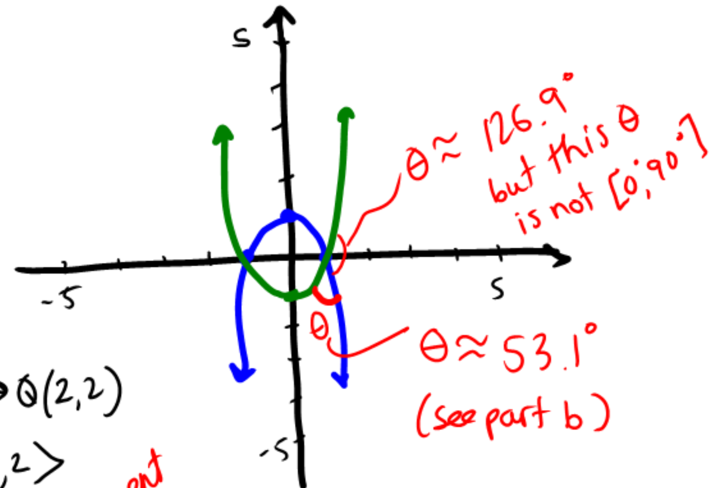
$x = \pm 1$

Find unit tangent vectors

$P(1,0) \to Q(2,-2)$

$\vec{u} = \langle 1, -2 \rangle$

$\frac{\vec{u}}{\|\vec{u}\|} = \frac{\langle 1, -2 \rangle}{\sqrt{5}}$ tangent to y_1



Find $\frac{\partial}{\partial x} y_1$, $\frac{\partial}{\partial x} y_2$ at $(1,0)$ and $(-1,0)$

$\frac{\partial}{\partial x} y_1 = \frac{\partial}{\partial x}(1 - x^2)$

$= -2x$

at $(1,0)$: $\frac{\partial y_1}{\partial x} = -2$

at $(-1,0)$: $\frac{\partial y_1}{\partial x} = 2$

$\frac{\partial}{\partial x} y_2 = \frac{\partial}{\partial x}(x^2 - 1)$

$= 2x$

at $(1,0)$: $\frac{dy_2}{dx} = 2$

at $(-1,0)$: $\frac{dy_2}{dx} = -2$

$P(1,0) \to Q(2,2)$

$\frac{\vec{u}}{\|\vec{u}\|} = \frac{\langle 1, 2 \rangle}{\sqrt{5}}$ tangent to y_2

$P(-1,0) \to Q(0,2)$

$\frac{\vec{u}}{\|\vec{u}\|} = \frac{\langle 1, 2 \rangle}{\sqrt{5}}$ tangent to y_1

$P(-1,0) \to Q(0,-2)$

$\frac{\vec{u}}{\|\vec{u}\|} = \frac{\langle 1, -2 \rangle}{\sqrt{5}}$ tangent to y_2

b. Find the angles ($0 \le \theta \le 90^\circ$) between the curves at their points of intersection.

At $(1,0)$, $\cos\theta = \frac{1}{\sqrt{5}}\langle 1, -2 \rangle \cdot \frac{1}{\sqrt{5}}\langle 1, 2 \rangle$

$\cos\theta = \frac{1}{5}\left[(1)(1) + (-2)(2)\right]$

$\cos\theta = \frac{-3}{5}$

$\theta = \arccos\frac{-3}{5}$

$\theta \approx 126.9^\circ$

ref ∡ $\theta \approx 53.1^\circ$

$\theta \approx 53.1^\circ$

DEFINITION OF UNIT TANGENT VECTOR

Let C be a smooth curve represented by $\mathbf{r}$ on an open interval I. The **unit tangent vector** $\mathbf{T}(t)$ at t is defined to be

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},\ \mathbf{r}'(t) \neq \mathbf{0}$$

The **tangent line to a curve** at a point is the line passing through point and parallel to the unit tangent vector.

Example 1: Find the unit tangent vector to the curve $\mathbf{r}(t) = e^t \cos t\mathbf{i} + e^t\mathbf{j}$ when $t = 0$.

$\vec{r}\,'(t) = \left[e^t \cos t + e^t(-\sin t)\right]\hat{i} + e^t\hat{j}$

$\vec{r}\,'(t) = e^t\left[(\cos t - \sin t)\hat{i} + \hat{j}\right]$

$\vec{r}\,'(0) = e^0\left[(\cos 0 - \sin 0)\hat{i} + \hat{j}\right]$

$\vec{r}\,'(0) = 1\left[(1 - 0)\hat{i} + \hat{j}\right]$

$\vec{r}\,'(0) = \hat{i} + \hat{j}$

$\|\vec{r}\,'(0)\| = \sqrt{2}$

$\boxed{\vec{T}(0) = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j})}$

Example 2: Consider the space curve $\mathbf{r}(t) = \left\langle t, t, \sqrt{4-t^2}\right\rangle$ at the point $(1, 1, \sqrt{3})$. a.

(x_0, y_0, z_0)

a. Find the unit tangent vector at the given point.

① Find t:

$x(t) = x_1 \rightarrow t = 1$

$y(t) = y_1 \rightarrow t = 1$

$z(t) = z_1 \rightarrow \sqrt{4-t^2} = \sqrt{3}$

$4 - t^2 = 3$

$1 = t^2$

$\cancel{\pm}1 = t$

② Find $\vec{T}(t)$

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$

$= \dfrac{\left\langle 1, 1, \frac{-\cancel{2}t}{\cancel{2}\sqrt{4-t^2}}\right\rangle}{\sqrt{(1)^2 + (1)^2 + \left(\frac{-t}{\sqrt{4-t^2}}\right)^2}}$

$= \dfrac{\left\langle 1, 1, -\frac{t}{\sqrt{4-t^2}}\right\rangle}{\sqrt{2 + t^2/(4-t^2)}}$

③ Find $\vec{T}(1)$

$\vec{T}(1) = \dfrac{\left\langle 1, 1, \frac{-(1)}{\sqrt{4-(1)^2}}\right\rangle}{\sqrt{2 + \frac{(1)^2}{4-(1)^2}}}$

$= \dfrac{\left\langle 1, 1, -\frac{1}{\sqrt{3}}\right\rangle}{\sqrt{\frac{7}{3}}}$

$= \frac{\sqrt{21}}{7}\left\langle 1, 1, -\frac{\sqrt{3}}{3}\right\rangle$

b. Find a set of parametric equations for the line tangent to the space curve at the given point.

$\vec{T}(1) = \frac{\sqrt{21}}{7}\left\langle 1, 1, -\frac{\sqrt{3}}{3}\right\rangle$

so $a = 1$
$b = 1$
$c = -\frac{\sqrt{3}}{3}$

$x = at + x_0 \rightarrow x = t + 1$

$y = bt + y_0 \rightarrow y = t + 1$

$z = ct + z_0 \rightarrow z = -\frac{\sqrt{3}}{3}t + \sqrt{3}$

DEFINITION: PRINCIPAL UNIT NORMAL VECTOR

Let C be a smooth curve represented by $\mathbf{r}$ on an open interval I. If $\mathbf{T}'(t) \neq \mathbf{0}$, then the **principal unit normal vector** $\mathbf{N}(t)$ at t is defined to be

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

At any point on a curve, a unit normal vector is orthogonal to the unit tangent vector. The principal unit normal vector points in the direction in which the curve is turning.

Example 3: Find the principal unit normal vector to the curve $\mathbf{r}(t) = \ln t\,\mathbf{i} + (t+1)\,\mathbf{j}$ at the time $t = 2$.

① Find $\vec{T}(t)$

$\vec{T}(t) = \dfrac{\frac{1}{t}\hat{\imath} + \hat{\jmath}}{\sqrt{\frac{1}{t^2} + 1}}$

$\vec{T}(t) = \dfrac{t^{-1}\hat{\imath} + \hat{\jmath}}{\sqrt{\frac{1+t^2}{t^2}}}$

$\vec{T}(t) = \dfrac{\hat{\imath} + t\hat{\jmath}}{\sqrt{1+t^2}}$

So $\vec{T}(t) = \left\langle (1+t^2)^{-1/2}, \frac{t}{\sqrt{1+t^2}} \right\rangle$

$\vec{T}(2) = \left\langle 5^{-1/2}, \frac{2}{\sqrt{5}} \right\rangle$

$\boxed{\vec{T}(2) = \frac{1}{\sqrt{5}}\langle 1, 2 \rangle}$

② Find $\vec{N}(t)$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

$$\vec{T}(t) = \left\langle (1+t^2)^{-1/2}, \frac{t}{\sqrt{1+t^2}} \right\rangle$$

$$T'(t) = \left\langle \frac{-t}{(1+t^2)^{3/2}}, \frac{1}{(1+t^2)^{3/2}} \right\rangle$$

$$\|T'(t)\| = \sqrt{\frac{t^2}{(1+t^2)^3} + \frac{1}{(1+t^2)^3}}$$

$$= \frac{(1+t^2)^{1/2}}{(1+t^2)^{3/2}} = \frac{1}{1+t^2}$$

$$\vec{N}(t) = (1+t^2)^{2/2} \left\langle -\frac{t}{(1+t^2)^{3/2}}, \frac{1}{(1+t^2)^{3/2}} \right\rangle$$

$$\vec{N}(t) = \frac{1}{\sqrt{1+t^2}} \langle -t, 1 \rangle$$

$$\vec{N}(2) = \frac{1}{\sqrt{1+(2)^2}} \langle -(2), 1 \rangle$$

$$\boxed{\vec{N}(2) = \frac{1}{\sqrt{5}} \langle -2, 1 \rangle}$$

$$\frac{\partial}{\partial t}\left((1+t^2)^{-1/2} \right)$$

$$= -\frac{1}{\cancel{2}} (1+t^2)^{-3/2} \cdot \cancel{2}t$$

$$= \frac{-t}{(1+t^2)^{3/2}}$$

$$\frac{\partial}{\partial t}\left(\frac{t}{(1+t^2)^{1/2}} \right)$$

$$= \frac{1(1+t^2)^{1/2} - t\left[\frac{1}{\cancel{2}}(1+t^2)^{-1/2} \cancel{2}t\right]}{1+t^2}$$

$$= \frac{(1+t^2)^{-1/2}\left[(1+t^2) - t^2\right]}{(1+t^2)^1}$$

$$= \frac{1}{(1+t^2)^{3/2}}$$

THEOREM: ACCELERATION VECTOR

If $\mathbf{r}(t)$ is the position vector for a smooth curve C and $\mathbf{N}(t)$ exists, then the acceleration vector $\mathbf{a}(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t)$ lies in the plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

THEOREM: TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

If $\mathbf{r}(t)$ is the position vector for a smooth curve C and $\mathbf{N}(t)$ exists, then the <u>**tangential and normal components of acceleration are as follows**</u>:

$$a_{\mathbf{T}} = D_t\left[\|\mathbf{v}\|\right] = \mathbf{a}\cdot\mathbf{T} = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|}$$

$$a_{\mathbf{N}} = \|\mathbf{v}\|\|\mathbf{T}'(t)\| = \mathbf{a}\cdot\mathbf{N} = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^{\,2}}$$

Note that $a_{\mathbf{N}} \geq 0$. The normal component of acceleration is also called the <u>**centripetal component of acceleration.**</u>

Example 4: Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$ for the plane curve $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k}$ at the time $t = 0$.

$x(t) = e^t \qquad y(t) = e^{-t} \qquad z(t) = t$

$\vec{a}(t) = \vec{r}\,''(t) = \langle e^t, e^{-t}, 0\rangle$

$\vec{a}(0) = \langle 1, 1, 0\rangle$

$\vec{r}\,'(t) = \langle e^t, -e^{-t}, 1\rangle = \vec{v}(t)$

$\|\vec{r}\,'(t)\| = \sqrt{e^{2t} + e^{-2t} + 1} = \sqrt{\dfrac{e^{4t} + 1 + e^{2t}}{e^{2t}}} = \dfrac{\sqrt{e^{4t} + e^{2t} + 1}}{e^t}$

$$\boxed{\vec{T}(t) = \frac{\langle e^{2t}, -1, e^t\rangle}{\sqrt{e^{4t} + e^{2t} + 1}}}$$

$\vec{T}(0) = \frac{1}{\sqrt{3}}\langle 1, -1, 1\rangle$

$$\frac{\partial}{\partial t}\left(\frac{e^{2t}}{(e^{4t} + e^{2t} + 1)^{1/2}}\right)$$

$$= \frac{2e^{2t}(e^{4t} + e^{2t} + 1)^{1/2} - e^{2t}\left[\dfrac{\cancel{4}^{\,2}e^{4t} + \cancel{2}^{\,1}e^{2t}}{\cancel{2}(e^{4t} + e^{2t} + 1)^{1/2}}\right]}{e^{4t} + e^{2t} + 1}$$

$$= \frac{e^{2t}(e^{4t} + e^{2t} + 1)^{-1/2}\left[(e^{4t} + e^{2t} + 1)\cdot 2 - (2e^{4t} + e^{2t})\right]}{(e^{4t} + e^{2t} + 1)^1}$$

$$= \frac{e^{2t}(e^{2t} + 2)}{(e^{4t} + e^{2t} + 1)^{3/2}}$$

$$\frac{d}{dt}\left(-\left(e^{4t}+e^{2t}+1\right)^{-1/2}\right) = \frac{1}{\cancel{2}}\left(e^{4t}+e^{2t}+1\right)^{-3/2}\cdot\left(\overset{2}{\cancel{4}}e^{4t}+\overset{1}{\cancel{2}}e^{2t}\right)$$

$$= \frac{2e^{4t}+e^{2t}}{\left(e^{4t}+e^{2t}+1\right)^{3/2}} \rightarrow e^{2t}\left(2e^{2t}+1\right)$$

$$\frac{d}{dt}\left(\frac{e^{t}}{\left(e^{4t}+e^{2t}+1\right)^{1/2}}\right) = \frac{e^{t}\left(e^{4t}+e^{2t}+1\right)^{1/2} - e^{t}\left(\frac{\overset{2}{\cancel{4}}e^{4t}+\overset{1}{\cancel{2}}e^{2t}}{\cancel{2}\left(e^{4t}+e^{2t}+1\right)^{1/2}}\right)}{e^{4t}+e^{2t}+1}$$

$$= \frac{e^{t}\left(e^{4t}+e^{2t}+1\right)^{-1/2}\left[\left(e^{4t}+e^{2t}+1\right)^{1}-\left(2e^{4t}+e^{2t}\right)\right]}{e^{4t}+e^{2t}+1}$$

$$= \frac{e^{t}\left(-e^{4t}+1\right)}{\left(e^{4t}+e^{2t}+1\right)^{3/2}}$$

magnitude

$$\sqrt{\left[\frac{e^{2t}\left(e^{2t}+2\right)}{\left(e^{4t}+e^{2t}+1\right)^{3/2}}\right]^2+\left[\frac{e^{2t}\left(2e^{2t}+1\right)}{\left(e^{4t}+e^{2t}+1\right)^{3/2}}\right]^2+\left[\frac{e^{t}\left(1-e^{4t}\right)}{\left(e^{4t}+e^{2t}+1\right)^{3/2}}\right]^2}$$

$$=\sqrt{\frac{e^{4t}\left(e^{2t}+2\right)^2+e^{4t}\left(2e^{2t}+1\right)^2+e^{2t}\left(1-e^{4t}\right)^2}{\left(e^{4t}+e^{2t}+1\right)^3}}$$

$$\vec{N}(t) = \frac{\cancel{\left(e^{4t}+e^{2t}+1\right)^{3/2}}\left\langle \frac{e^{2t}\left(e^{2t}+2\right)}{\cancel{\left(e^{4t}+e^{2t}+1\right)^{3/2}}}, \frac{e^{2t}\left(2e^{2t}+1\right)}{\cancel{\left(e^{4t}+e^{2t}+1\right)^{3/2}}}, \frac{e^{t}\left(1-e^{4t}\right)}{\cancel{\left(e^{4t}+e^{2t}+1\right)^{3/2}}}\right\rangle}{\left[e^{4t}\left(e^{2t}+2\right)^2+e^{4t}\left(2e^{2t}+1\right)^2+e^{2t}\left(1-e^{4t}\right)^2\right]^{1/2}}$$

$$\vec{N}(0) = \frac{\left\langle e^0(e^0+2),\ e^0(2e^0+1),\ e^0(1-e^0) \right\rangle}{\left[e^0(e^0+2)^2 + e^0(2e^0+1)^2 + e^0(1-e^0)^2\right]^{\frac{1}{2}}}$$

$$\vec{N}(0) = \frac{\langle 3, 3, 0 \rangle}{\sqrt{9+9+0}}$$

$$\vec{N}(0) = \frac{\langle 3,3,0 \rangle}{3\sqrt{2}}$$

$$\vec{N}(0) = \frac{1}{\sqrt{2}}\langle 1, 1, 0 \rangle$$

at $t=0$,

$$a_{\vec{T}} = \vec{a}(0) \cdot \vec{T}(0)$$

$$= \langle 1,1,0 \rangle \cdot \frac{1}{\sqrt{3}}\langle 1,-1,1 \rangle$$

$$= \frac{1}{\sqrt{3}}\left[(1)(1) + (1)(-1) + (0)(1)\right]$$

$$= \frac{1}{\sqrt{3}} \cdot 0$$

$$= \boxed{0}$$

$$a_{\vec{N}} = \vec{a}(0) \cdot \vec{N}(0)$$

$$= \langle 1,1,0 \rangle \cdot \frac{1}{\sqrt{2}}\langle 1,1,0 \rangle$$

$$= \frac{1}{\sqrt{2}}(1+1+0)$$

$$= \frac{2}{\sqrt{2}}$$

$$= \boxed{\sqrt{2}}$$