LINEAR SYSTEMS, MATRICES, AND VECTORS

Now that I've been teaching Linear Algebra for a few years, I thought it would be great to integrate the more advanced topics such as vector spaces, the Euclidean dot product, and matrix operations early on in our class, instead of hurrying to fit everything in late in the course. So…hold on to your seats...we're in for a bumpy ride!

1.1 Linear Systems and Matrices

Learning Objectives

1. Use back-substitution and Gaussian elimination to solve a system of linear equations
2. Determine whether a system of linear equations is consistent or inconsistent
3. Find a parametric representation of a solution set
4. Write an augmented or coefficient matrix from a system of linear equations
5. Determine the size of a matrix

Let's Do Our Math Stretches!

1. Solve the following systems of linear equations
    a.
    $$\begin{aligned} -x+8y &= 3 \\ 6x &= 12 \end{aligned}$$
    b.
    $$\begin{aligned} 3x+y-z &= 15 \\ 2y+4z &= 0 \\ z &= 1 \end{aligned}$$

DEFINITION OF A LINEAR EQUATION IN *n* VARIABLES

A linear equation in *n* variables ________________________ has the form

The _________________ $a_1, a_2, a_3, \ldots, a_n$ are ___________ numbers, and the ______________ term b is a real number. The number a_1 is the _________________ _____________________, and ______ is the leading variable.

*Linear equations have no _________________ or ____________ of variables and no variables involved in ________________________________ functions.

Example 1: Give an example of a linear equation in three variables.

DEFINITION OF SOLUTIONS AND SOLUTION SETS

A solution of a linear equation in *n* variables is a ___________________ of *n* real numbers $s_1, s_2, s_3, \ldots, s_n$ arranged to satisfy the equation when you substitute the values

into the equation. The set of _______ solutions of a linear equation is called its _______________ ________, and when you have found this set, you have _________________ the equation. To describe the entire solution set of a linear equation, use a ____________________ representation.

Example 2: Solve the linear equation.

$x_1 + x_2 = 10$

Example 3: Solve the linear equation.

$2x_1 - x_2 + 5x_3 = -1.$

SYSTEMS OF LINEAR EQUATIONS IN *n* VARIABLES

A system of linear equations in *n* variables is a set of *m* equations, each of which is linear in the same *n* variables.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

SOLUTIONS OF SYSTEMS OF LINEAR EQUATIONS

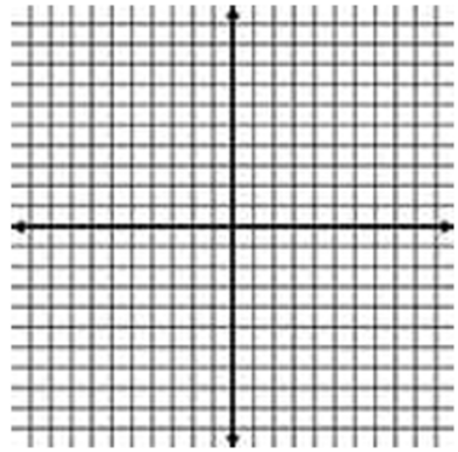
A solution of a system of linear equations is a ____________________ of numbers $s_1, s_2, s_3, \ldots, s_n$ that is a solution of each of the linear equations in the ______________________.

Example 4: Graph the following linear systems and determine the solution(s), if a solution exists.

a.

$x - y = 8$

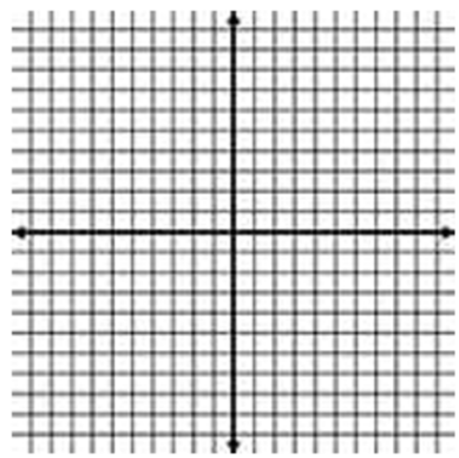
$x + y = 2$



b.

$x - y = 8$

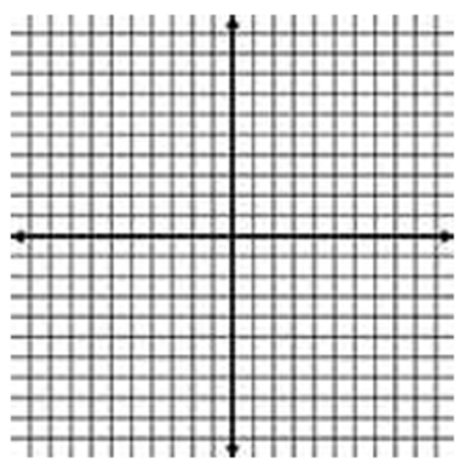
$x - y = 2$



c.

$2x - 2y = 16$

$3x - 3y = 6$



NUMBER OF SOLUTIONS OF A SYSTEM OF EQUATIONS

For a system of linear equations, precisely one of the following is true.

The system has _______________ __________ solution. (_________________ system).

The system has _________________ ___________ solutions (_________________ system)

The system has _______ _______________ (_______________ system).

TYPES OF SOLUTIONS

2 Equations, 2 Variables

What did we learn from the last example?

Inconsistent:

Consistent:

3 Equations, 3 Variables

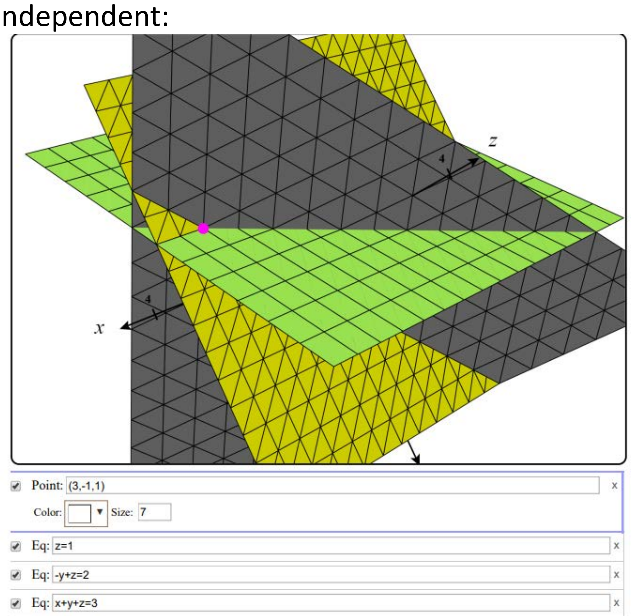
Inconsistent

Parallel Planes   Intersecting Two at a Time (1) or Intersecting Two at a Time (2)

Consistent

Dependent: Linear Intersection   Planar Intersection

Independent:



OPERATIONS THAT PRODUCE EQUIVALENT SYSTEMS

Each of the following operations on a system of linear equations produces an ______________________ system.

___________________ two equations.

___________________ an equation by a __________________ constant.

_________ a ________________ of an equation to _________________ equation.

The evil plan is to get the system into ______________________________ form.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{33}x_3 &= b_3 \end{aligned}$$

DEFINITION OF A MATRIX

If *m* and *n* are positive integers, an $m \times n$ matrix (read ______________) matrix is a _____________________ array

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

in which each _____________, a_{ij}, of the matrix is a number. An $m \times n$ matrix has m rows and n columns.

Matrices are usually denoted by _______________ letters.

*The entry a_{ij} is located in the *i*th row and the *j*th column. The index *i* is called the ________ ___________________ because it identifies the row in which the entry lies, and the index *j* is called the

__________________ _____________ because it identifies the column in which the entry lies.

**A matrix with *m* rows and *n* columns is said to be of ___________ _________. When _______________, the matrix is called ____________ of order *n* and the entries $a_{11}, a_{22}, a_{33}, \ldots$ are called the ____________

__________________ entries.

THREE IMPORTANT TYPES OF MATRICES

1. _______________ Matrices are square matrices with ___________ along the main _______________, and zeros ___________________. The main diagonal goes from the top ___________ corner to the ___________________ right corner.
2. _____________________ Matrices are formed using the ___________________ of the ____________ in systems of linear equations.
3. _______________ Matrices adjoin the coefficient matrix with the column matrix of _______________.

Example 5: Consider the following system of linear equations.

$$\begin{aligned} x_1 - x_2 \quad + x_3 &= 2 \\ -x_1 + 3x_2 - 2x_3 &= 8 \\ 2x_1 + x_2 \quad - x_3 &= 1 \end{aligned}$$

a. Find the coefficient matrix (matrix of coefficients) and determine its size.

b. Find the augmented matrix and determine its size.

c. Solve the system and determine if it is consistent.

d. Check your result using Octave, which has the same commands as Matlab but is free☺.

i. Go to the very bottom of the page and enter the augmented matrix. I named the augmented matrix B. You use brackets to designate a matrix, use a _______________ between entries, and a ___________________________ between rows.

```
》B = [1 -1 1 2;-1 3 -2 8; 2 1 -1 1]
```

ii. After hitting "enter" the screen looks like this (you'll have a different command line number):

```
octave:18> B = [1 -1 1 2;-1 3 -2 8; 2 1 -1 1]
B =

   1  -1   1   2
  -1   3  -2   8
   2   1  -1   1
```

Now type in rref(B) to get the reduced row-echelon form of the augmented matrix:

```
octave:18> B = [1 -1 1 2;-1 3 -2 8; 2 1 -1 1]
B =

   1  -1   1   2
  -1   3  -2   8
   2   1  -1   1

》rref(B)
```

After hitting enter, you'll see:

```
octave:18> B = [1 -1 1 2;-1 3 -2 8; 2 1 -1 1]
B =

   1  -1   1   2
  -1   3  -2   8
   2   1  -1   1

octave:19> rref(B)
ans =

    1.00000    0.00000    0.00000    1.00000
    0.00000    1.00000    0.00000   11.00000
    0.00000    0.00000    1.00000   12.00000
```

iii. How should we interpret the results?

1.2 Gauss-Jordan Elimination

Learning Objectives

1. Use matrices and Gaussian elimination with back-substitution to solve a system of linear equations
2. Use matrices and Gauss-Jordan elimination to solve a system of linear equations
3. Solve a homogeneous system of linear equations
4. Fit a polynomial function to a set of data points
5. Set up and solve a system of equations to represent a network

Let's Do Our Math Stretches!

1. Interpret the following **augmented** matrices.

a.

$$\begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

b.

$$\begin{bmatrix} 1 & -1 & 0 & -2 \\ 0 & 1 & 3 & 11 \end{bmatrix}$$

c.

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 3 \\ 0 & 1 & 0 & 7 & 0 \end{bmatrix}$$

ELEMENTARY ROW OPERATIONS

1. ___________________ two rows.

2. ___________________ a row by a __________________ constant.

3. _________ a ________________ of a row to _________________ row.

4. ____________________ (_________) any 2 rows.

Note: These operations also work for columns.

DEFINITION OF ROW-ECHELON FORM AND REDUCED ROW-ECHELON FORM

A matrix in ____________________________ form has the following properties.

Any rows consisting entirely of _____________ occur at the bottom of the matrix. For each row that does not consist entirely of zeros, the first nonzero entry is ______ (called a leading ____). For two successive nonzero rows, the leading 1 in the higher row is farther to the _________ than the leading 1 in the lower row. A matrix in row-echelon form is in ________ ____________________ form when every column that has a leading 1 has ______________ in every position above and below its leading 1.

Example 1: Determine which of the following augmented matrices are in row-echelon (ref) form.

a.
$$\begin{bmatrix} 1 & -\frac{1}{2} \end{bmatrix}$$

b.
$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 12 \end{bmatrix}$$

c.
$$\begin{bmatrix} 1 & 1 & -1 & -8 \\ 0 & 0 & 1 & 25 \\ 0 & 1 & 15 & -3 \end{bmatrix}$$

GAUSS-JORDAN ELIMINATION

1. Write the _____________________ matrix of the system of linear equations.

2. Use elementary row operations to find an_____________________ matrix in _____________________ row-echelon form. If this is not possible, write the equivalent system of equations and back substitute.

3. Interpret your results.

Example 2: Solve the system using Gauss-Jordan Elimination.

a.

$$\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 - 2x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$$

b.

$$\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 3 \\ 2x_1 + 4x_2 - x_3 &= 7 \\ x_1 - 11x_2 + 4x_3 &= 3 \end{aligned}$$

DEFINITION OF HOMOGENEOUS SYSTEMS OF LINEAR EQUATIONS

Systems of equations in which each of the ________________ terms is zero are called

_____________________. A homogeneous system of *m* equations in *n* variables has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots a_{2n}x_n &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots a_{3n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots a_{mn}x_n &= 0 \end{aligned}$$

****Homogenous linear systems either have the ______________ solution, or _________________**

_______________ solutions

Example 3: Solve the homogeneous linear system corresponding to the given coefficient matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

THEOREM 1.1: THE NUMBER OF SOLUTIONS OF A HOMOGENEOUS SYSTEM

Every homogeneous system of linear equations is ____________________. If the system has fewer equations than variables, then it must have _______________ __________________ solutions.

POLYNOMIAL CURVE FITTING

Suppose *n* points in the *xy*-plane represent a collection of ____________ and you are asked to find a _________________________________ function of degree ____________ whose graph passes through the specified points. This is called _________________________ ________________ ______________________. If all *x*-coordinates are distinct, then there is precisely _______ polynomial function of degree *n* – 1 (or less) that fits the *n* points. To solve for the *n* ______________________ of $p(x)$, ___________________ each of the *n* points into the polynomial function and obtain *n* __________________ equations in ____ variables $a_0, a_1, a_2, \ldots, a_{n-1}$.

$$\begin{aligned} a_0 + a_1x_1 + a_2x_1{}^2 + \cdots a_{n-1}x_1{}^{n-1} &= y_1 \\ a_0 + a_1x_2 + a_2x_2{}^2 + \cdots a_{n-1}x_2{}^{n-1} &= y_2 \\ a_0 + a_1x_3 + a_2x_3{}^2 + \cdots a_{n-1}x_3{}^{n-1} &= y_3 \\ &\vdots \\ a_0 + a_1x_n + a_2x_n{}^2 + \cdots a_{n-1}x_n{}^{n-1} &= y_n \end{aligned}$$

Example 4: Determine the polynomial function whose graph passes through the points, and graph the polynomial function, showing the given points.

$(1,8),(3,26),(5,60)$

NETWORK ANALYSIS

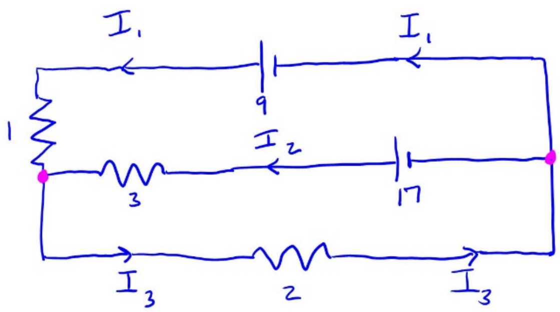
Networks composed of ____________________ and ___________________ are used as models in fields like economics, traffic analysis, and electrical engineering. In an electrical network model. you use Kirchoff's Laws

to find the system of equations.

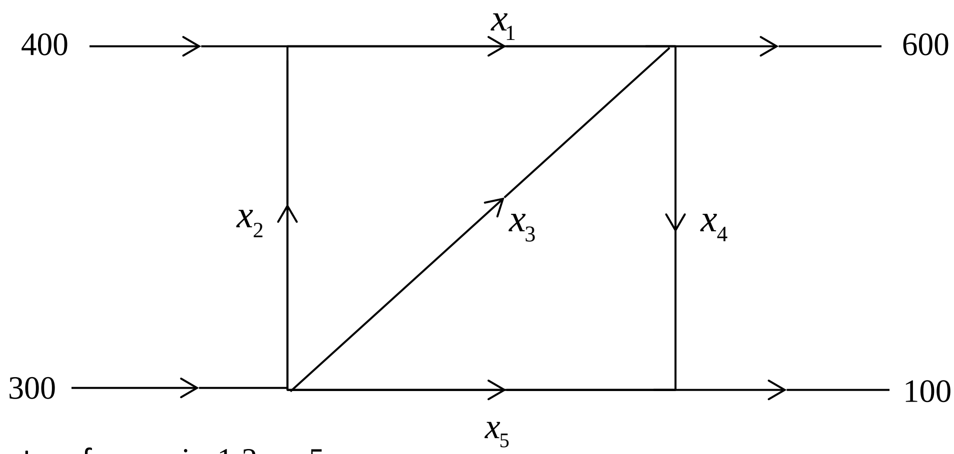
Kirchoff's Laws

1. Junctions: All the current flowing into a junction must flow out of it.
2. Paths: The sum of the *IR* terms, where *I* denotes _______________ and *R* denotes ______________, in any direction around a closed path is equal to the total voltage in the path in that direction.

Example 5: Determine the currents in the various branches of the electrical network. The units of current are amps and the units of resistance are ohms.



Example 6: The figure below shows the flow of traffic through a network of streets.



Solve this system for x_i , $i = 1, 2, \ldots, 5$.

Find the traffic flow when $x_3 = 0$ and $x_5 = 100$.

Find the traffic flow when $x_3 = x_5 = 100$.

1.3 The Vector Space R^n

Learning Objectives

1. Perform basic vector operations in R^2 and represent them graphically
2. Perform basic vector operations in R^n
3. Write a vector as a linear combination of other vectors
4. Perform basic operations with column vectors
5. Determine whether one vector can be written as a linear combination of 2 or more vectors
6. Determine if a subset of R^n is a subspace of R^n

VECTORS IN THE PLANE

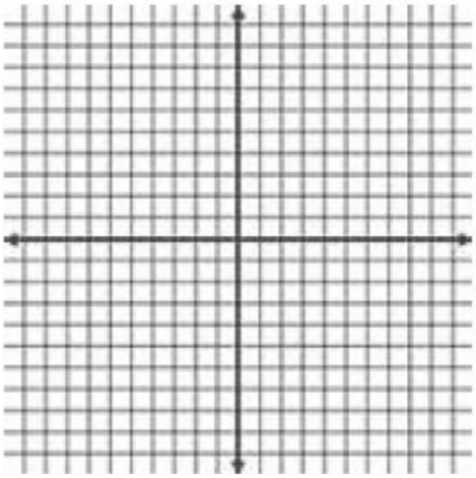
A vector is characterized by two quantities, __________________ and __________________, and is represented by a ________________ ________ _________________. Geometrically, a _______________ in the __________ is represented by a directed line segment with its ____________ ___________ at the origin and its _____________ point at _______________. Boldface lowercase letters often designate ____________ when you're using a computer, but when you write them by hand you need to write an ______________ above the letter designating the vector.

The same _______________ ____________ used to represent its terminal point also represents the ________________. That is, ________________. The coordinates x_1 and x_2 are called the ______________________ of the vector $\mathbf{x}$. Two vectors in the plane $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are ___________ if and only if ________________ and _________________. What do you think the zero vector is for R^2 ? ________________ How about R^3 ? ________________ R^6 ?______________________________

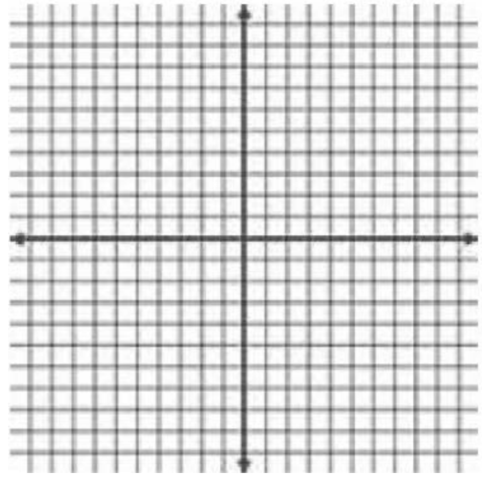
R^n ?________________________________

Example 1: Use a directed line segment to represent the vector, and give the graphical representation of the vector operations.

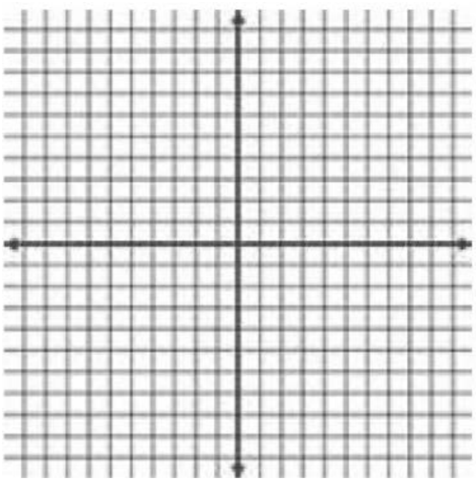
a. $\mathbf{u} = (3, -2)$



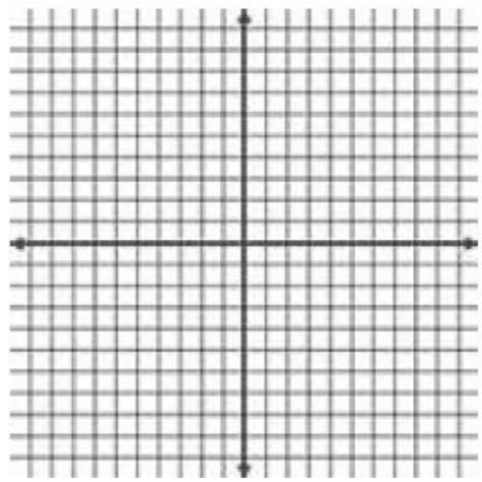
b. $\mathbf{v} = (2, 1)$



c. $\mathbf{u} + \mathbf{v}$



d. $-2\mathbf{v}$



IMPORTANT VECTOR SPACES

_______ = ______________________ = the set of _________________ _____________.
________ = ________________________ = the set of all ____________________ _____________of real numbers.
________ = ________________________ = the set of all ____________________ _____________of real numbers.
________ = ________________________ = the set of all ____________________ _____________of real numbers.

DEFINITION OF VECTOR ADDITION AND SCALAR MULTIPLICATION

Let ________________________ and ____________________________ be vectors in _____, and let ____________.
Then the sum of _______ and ______ is defined as the _________________, ______________________________,
and the ________________ multiplication of ______ by ______ is defined as the ________________,
___________________________.

THEOREM 1.2: PROPERTIES OF VECTOR ADDITION AND SCALAR MULTIPLICATION IN R^n

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in R^n, and let c and d be scalars.
ADDITION:

1. $\mathbf{u}+\mathbf{v}$ is a ____________ in R^n. ____________________

Proof:

2. $\mathbf{u}+\mathbf{v}=$ __________________ ____________________ property

Proof:

3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=$ __________________ ____________________ property
4. $\mathbf{u}+\mathbf{0}=$ __________________ additive ______________ property
5. $\mathbf{u}+(-\mathbf{u})=$ __________________ additive ______________ property

SCALAR MULTIPLICATION:

6. $c\mathbf{u}$ is a _________ in the R^n. ____________________
7. $c(\mathbf{u}+\mathbf{v})=$ __________________ ____________________ property

Proof:

8. $(c+d)\mathbf{u}=$ __________________ ____________________ property
9. $c(d\mathbf{u})=$ __________________ ____________________ property
10. $1(\mathbf{u})=$ __________________ _________________ ______________ property

Example 2: Solve for $\mathbf{w}$, where $\mathbf{u} = (2, -1, 3, 4)$, and $\mathbf{v} = (-1, 8, 0, 3)$.

a. $\mathbf{w} + \mathbf{u} = -\mathbf{v}$

b. $\mathbf{w} + 3\mathbf{v} = -2\mathbf{u}$

DEFINITION OF COLUMN VECTOR ADDITION AND SCALAR MULTIPLICATION

Let $u_1, u_2, \ldots, u_n$, $v_1, v_2, \ldots, v_n$, and c be scalars.

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} \text{ and } c\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$$

Example 3: Find the following, given that $\mathbf{u} = \begin{bmatrix} -3 \\ 18 \\ -1 \\ 31 \\ -9 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -2 \\ 41 \\ -6 \\ -3 \\ 15 \end{bmatrix}$.

a. $2\mathbf{u} - 3\mathbf{v}$

b. $-(\mathbf{v} + \mathbf{u})$

THEOREM 1.3: PROPERTIES OF ADDITIVE IDENTITY AND ADDITIVE INVERSE

Let $\mathbf{v}$ be a vector in R^n, and let c be a scalar. Then the following properties are true.

1. The ______________ ________________ is _________________.

Proof:

2. The ______________ ________________ is _________________.
3. $0\mathbf{v} =$ ___________
4. $c\mathbf{0} =$ ___________
5. If $c\mathbf{v} = \mathbf{0}$, then _________________ or ____________________.
6. $-(-\mathbf{v}) =$ ___________

LINEAR COMBINATIONS OF VECTORS

An important type of problem in linear algebra involves writing one vector as the _________ of ______________ ____________________ of other vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The vector ____,

____________________________ is called a _____________ ________________ of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Example 4: If possible, write $\mathbf{u}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1 = (1,2)$ and $\mathbf{v}_2 = (-1,3)$.

a. $\mathbf{u} = (0,3)$

b. $\mathbf{u} = (1,-1)$

Example 5: If possible, write $\mathbf{u}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, where $\mathbf{v}_1 = (1,3,5)$, $\mathbf{v}_2 = (2,-1,3)$, and $\mathbf{v}_3 = (-3,2,-4)$.

$\mathbf{u} = (-1,7,2)$

WHAT THE HECK DOES IT ALL MEAN??

Any vector space consists of _______________ entities: a _______ of _____________, a set of __________________, and ___________ operations. Currently, we are only exploring the vector space, ______.

Let's think about the following subset of R^2:

$$S = \left\{ \left(x, \frac{1}{2}x \right) : x \in \mathbb{R} \right\}$$

Is the set S a vector space? Let's find out!

1. Closure under addition.

2. Commutativity under addition.

3. Associativity under addition.

4. Additive identity.

5. Additive inverse.

6. Closure under scalar multiplication.

7. Distributivity under scalar multiplication (2 vectors and 1 scalar).

8. Distributivity under scalar multiplication (2 scalars and 1 vector).

9. Associativity under scalar multiplication.

10. Scalar multiplicative identity.

Conclusion?

Example 6: Determine whether the set W is a vector space with the standard operations. Justify your answer.

$W = \{(x_1, x_2, 4) : x_1 \text{ and } x_2 \in \mathbb{R}\}$

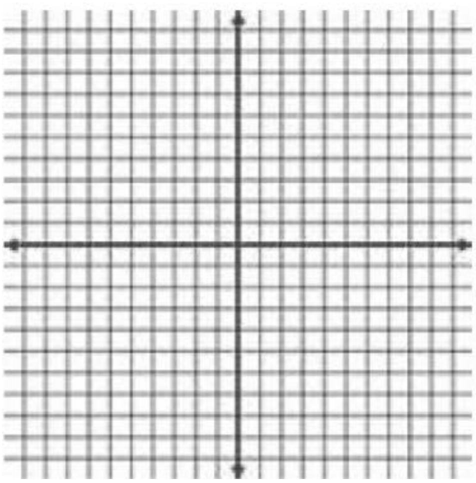
SUBSPACES

In many applications of linear algebra, vector spaces occur as a ______________ of larger spaces. A

__________________ subset of a vector _____________ is a _________________ when it is a vector

__________________ with the ____________ operations defined in the ________________ vector space.

Consider the following: $W = (0, y)$ and $V = R^2$.



DEFINITION OF A SUBSPACE OF A VECTOR SPACE

A nonempty subset W of a vector space V is called a ___________________ of V when _______ is a vector space under the operations of _________________ and ________________ ______________________________ defined in V.

THEOREM 1.4: TEST FOR A SUBSPACE

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following closure conditions hold.

1. If $\mathbf{u}$ and $\mathbf{v}$ are in W, then ______________ is in W.
2. If $\mathbf{u}$ is in W and c is any scalar, then __________ is in W.

Example 7: Verify that W is a subspace of V.

$W = \{(x, y, 2x-3y) : x \text{ and } y \in \mathbb{R}\}$

$V = R^3$

THEOREM 1.5: THE INTERSECTION OF TWO SUBSPACES IS A SUBSPACE

If V and W are both subspaces of a vector space U, then the intersection of V and W, denoted by ______________________, is also a subspace of U.

1.4 Basis and Dimension of R^n

Learning Objectives

1. Determine if a set of vectors in R^n spans R^n.
2. Determine if a set of vectors in R^n is linearly independent
3. Determine if a set of vectors in R^n is a basis for R^n
4. Find standard bases for R^n
5. Determine the dimension of R^n

Let's do our math stretches!

If possible, write the vector $\mathbf{z} = \left(-4, -3, 3\right)$ as a linear combination of the vectors in $S = \left\{\left(1, 2, -2\right), \left(2, -1, 1\right)\right\}$.

DEFINITION OF LINEAR COMBINATION OF VECTORS IN A VECTOR SPACE

A vector $\mathbf{v}$ in a vector space V is called a ____________________ combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in V

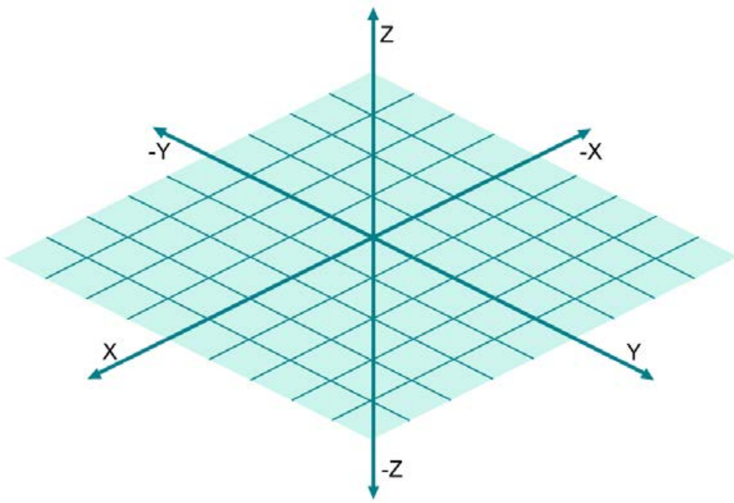
when $\mathbf{v}$ can be written in the form

where $c_1, c_2, \ldots c_k$ are scalars.

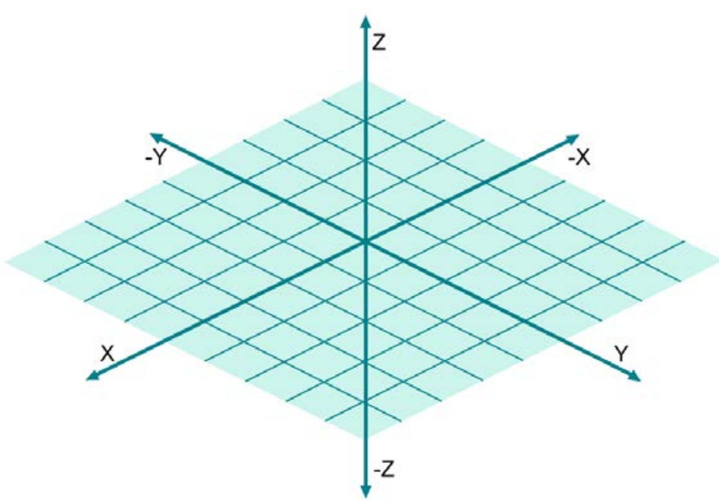
DEFINITION OF A SPANNING SET OF A VECTOR SPACE

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ be a subset of a vector space V. The set S is called a ________________ set of V when __________ vector in V can be written as a _________________ ______________________ of vectors in S.

$S = \{(1,0,0),(0,1,0),(0,0,1)\}$, $V = R^3$



$S = \{(1,2,3),(0,1,2),(-1,1,1)\}$, $V = R^3$



DEFINITION OF THE SPAN OF A SET

If $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$ is a set of vectors in a vector space V, then the ____________ of S is the set of all

________________ combinations of the vectors in S.

The span of S is denoted by

When _________________, it is said that V is _________________ by _______________________, or that

_____ spans _____.

THEOREM 1.6: Span(S) IS A SUBSPACE OF V

If $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$ is a set of a vectors in a vector space V, then $\operatorname{span}(S)$ is a subspace of V. Moreover, $\operatorname{span}(S)$ is the ___________________ subspace of V that contains S, in the sense that every other subspace of V that contains S must contain $\operatorname{span}(S)$.

Proof:

Example 3: Determine whether the set S spans R^2. If the set does not span R^2, then give a geometric description of the subspace that it does span.

a. $S = \{(1,-1),(2,1)\}$

b. $S = \left\{(1,2),(-2,-4),\left(\frac{1}{2},1\right)\right\}$

c. $S=\{(-1,2),(2,-1),(1,1)\}$

DEFINITION OF LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

A set of vectors $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$ in a vector space V is called linearly ________________________ when the vector equation

has only the ____________ solution

If there are also ___________________ solutions, then S is called linearly ___________________________.

TESTING FOR LINEAR INDEPENDENCE AND LINEAR DEPENDENCE

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ be a set of vectors in a vector space V. To determine whether S is linearly independent of linearly dependent, use the following steps.

1. From the vector equation _________________________________, write a ____________________ of linear equations in the variables $c_1, c_2, ...,$ and c_k.
2. Use Gaussian elimination to determine whether the system has a __________________ solution.
3. If the system has only the __________________ solution, $c_1 = 0, c_2 = 0, ..., c_k = 0$, then the set S is linearly independent. If the system has ____________________ solutions, then S is linearly dependent.

Example 4: Determine whether the set S is linearly independent or linearly dependent.

a. $S = \{(3, -6), (-1, 2)\}$

b. $S = \{(6, 2, 1), (-1, 3, 2)\}$

c. $S=\{(0,0,0,1),(0,0,1,1),(0,1,1,1),(1,1,1,1)\}$

THEOREM 1.7: A PROPERTY OF LINEARLY DEPENDENT SETS

A set $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$, $k\geq 2$, is linearly dependent if and only if at least one of the vectors $\mathbf{v}_j$ can be written as a linear combination of the other vectors in S.

Proof:

THEOREM 1.7: COROLLARY

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in a vector space V are linearly dependent if and only if one is a ____________________ of the other.

Example 5: Show that the set is linearly dependent by finding a nontrivial linear combination of vectors in the set whose sum is the zero vector. Then express one of the vectors in the set as a linear combination of the other vectors in the set.

$S = \{(2,4),(-1,-2),(0,6)\}$

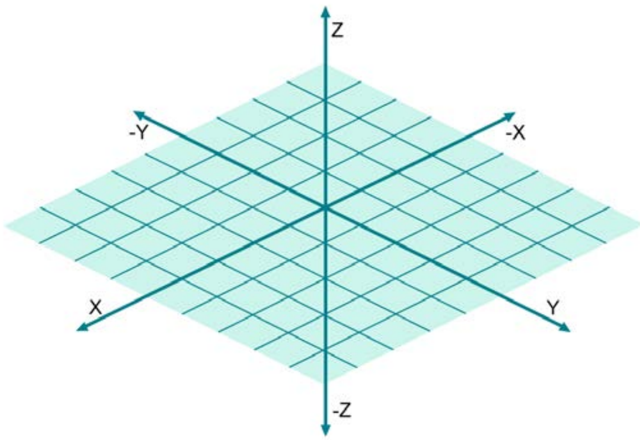
DEFINITION OF BASIS

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is called a ___________________ for _______ when the following conditions are true.

1. S __________ V.
2. S is linearly __________________.

The Standard Basis for R^3

$S = \{(1,0,0),(0,1,0),(0,0,1)\}$



Example 6: Write the standard basis for the vector space.

a. R^2

b. R^5

c. R^n

Example 7: Determine whether S is a basis for the indicated vector space.

$S = \{(2,1,0),(0,-1,1)\}$ for R^3

THEOREM 1.8: UNIQUENESS OF BASIS REPRESENTATION

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.

Proof:

THEOREM 1.9: BASES AND LINEAR DEPENDENCE

If $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a basis for a vector space V, then every set containing more than ______ vectors in V is linearly ______________________.

THEOREM 1.10: NUMBER OF VECTORS IN A BASIS

If a vector space V has one basis with ____ _______________, then every basis for V has ____ vectors.

Proof:

DEFINITION OF DIMENSION OF A VECTOR SPACE

If a vector space V has a _________________ consisting of _____ vectors, then the number _____ is called the _____________________ of V, denoted by ________________________. When V consists of the ___________ vector alone, the dimension of V is defined as _______.

Example 8: Determine the dimension of the vector space.

a. R^2

b. R^5

c. R^n

THEOREM 1.11: BASIS TESTS IN AN n-DIMENSIONAL SPACE

Let V be a vector space of dimension n.

1. If ___________________ is a linearly independent set of vectors in V, then ______ is a ______________ for _______.

2. If ___________________ __________ V, then ______ is a _____________ for ________.

Example 9: Determine whether S is a basis for the indicated vector space.

$S = \{(1,2),(1,-1)\}$ for R^2.

2.1 Matrix Operations

Learning Objectives

1. Determine whether two matrices are equal
2. Add and subtract matrices, and multiply a matrix by a scalar
3. Multiply two matrices
4. Use matrices to solve a system of equations
5. Partition a matrix and write a linear combination of column vectors

Matrices can be thought of as adjoined column vectors. They are represented in the following ways:

1. ____________________ letter

2. Representative ___________________

3. Rectangular _____________________

DEFINITION OF EQUALITY OF MATRICES

Two matrices $A = \left[a_{ij} \right]$ and $B = \left[b_{ij} \right]$ are _______________ when they have the same ________________ _______________ and ____________________ for _______________ and ___________________.

Example 1: Are matrices A and B equal? Please explain.

$$A = \begin{bmatrix} 1 & -1 & 3 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 8 \end{bmatrix}$$

Example 2: Find x and y.

$$\begin{bmatrix} 2x-1 & 4 \\ 3 & y^3 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 3 & \frac{1}{8} \end{bmatrix}$$

A matrix that has only one ________________ is called a _________________ __________________ or __________________ ______________________. A matrix that has only one ________________ is called a ________________ __________________ or __________________ ______________________. As we learned earlier, boldface lowercase letters often designate _________________ ___________ and ___________________ __________________.

DEFINITION OF MATRIX ADDITION

If $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ are matrices of size $m \times n$, then their __________ is the $m \times n$ matrix given by

The sum of two matrices of different sizes is ________________________.

DEFINITION OF SCALAR MULTIPLICATION

If $A=\left[a_{ij}\right]$ is an $m \times n$ matrix and c is a scalar, then the ________________ ____________________ of A by c is the ______________ matrix given by

Note: You can use _______ to represent the scalar product _________. If A and B are of the same size, then $A-B$ represents the sum of _____ and ________.

Example 3: Find the following for the matrices

$$A=\begin{bmatrix}1 & -3 & 6\\ 2 & 0 & 2\\ -2 & 8 & -1\end{bmatrix} \text{ and } B=\begin{bmatrix}5 & 2 & 7\\ -1 & 9 & -4\\ -3 & 0 & 1\end{bmatrix}$$

a. $A+B$

b. $2A-B$

DEFINITION OF MATRIX MULTIPLICATION

If $A=\left[a_{ij}\right]$ is an $m\times n$ matrix and $B=\left[b_{ij}\right]$ is an $n\times p$ matrix, then the _______________ AB is an $m\times p$ matrix.

where

To find an entry in the *i*th row and the *j*th column of the product AB, multiply the ______________ in the __________ row of A by the corresponding entries in the ___________ column of B and then ________ the results.

Example 4: Find the product AB, where

$A = \begin{bmatrix} 15 & 0 \\ 4 & 5 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -12 & 7 & 5 & -1 \\ -13 & 1 & 2 & 11 \end{bmatrix}$

Example 5: Consider the matrices A and B.

$A = \begin{bmatrix} -1 & 3 \\ 11 & 13 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 4 \\ 6 & 13 \end{bmatrix}$

a. Find $A+B$

b. Find $B+A$

c. Find AB

d. Find BA

Is matrix addition commutative?

Is matrix multiplication commutative?

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

SYSTEMS OF LINEAR EQUATIONS

The system

can be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

or equivalently,

Example 6: Write the system of equations in the form $A\mathbf{x} = \mathbf{b}$ and solve this matrix equation for $\mathbf{x}$.

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ x_1 + 4x_2 &= 10 \end{aligned}$$

PARTITIONED MATRICES

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

LINEAR COMBINATIONS (MATRICES)

The matrix product $A\mathbf{x}$ is a linear combination of the _______________ vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_n$ that form the

__________________________ matrix A.

The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ can be expressed as such a ____________________

_________________________________, where the ______________________ of the linear combination are a

______________________ of the system.

Example 7: Write the column matrix $\mathbf{b}$ as a linear combination of the columns of A

$$A = \begin{bmatrix} -1 & 3 \\ 16 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -7 \\ 63 \end{bmatrix}$$

Example 8: Find the products AB and BA for the diagonal matrices.

$$A=\begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad B=\begin{bmatrix} -7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

Example 9: Use the given partitions of A and B to compute AB.

$$A=\left[\begin{array}{c|c} 2 & 1 \\ -1 & 0 \\ \hline 3 & 1 \end{array}\right], \text{ and } B=\left[\begin{array}{cc} 3 & 0 \\ \hline 2 & 1 \end{array}\right]$$

2.2: Properties of Matrix Operations

Learning Objectives

1. Use the properties of matrix addition, scalar multiplication, and zero matrices
2. Use the properties of matrix multiplication and the identity matrix
3. Find the transpose of a matrix
4. Use Stochastic matrices for applications

THEOREM 2.1: PROPERTIES OF MATRIX ADDITION AND SCALAR MULTIPLICATION

If A, B, and C are $m \times n$ matrices, and c and d are scalars, then the following properties are true.

1. $A+B=$ ________________________ Commutative property of addition

Proof:

2. $A+(B+C)=$ ____________________ Associative property of addition

3. $(cd)A=$ ________________________ Associative property of multiplication

4. $1A=$ ___________________________ Multiplicative Identity

5. $c(A+B)=$ _____________________Distributive property

Proof:

6. $(c+d)A=$ _____________________ Distributive property

Example 1: For the matrices below, $c=-2$, and $d=5$,

$$A=\begin{bmatrix}-3 & 5\\ 3 & 4\\ 4 & 8\end{bmatrix} \qquad B=\begin{bmatrix}1 & 1\\ 2 & 7\\ 6 & 9\end{bmatrix} \qquad C=\begin{bmatrix}-7 & 1\\ -2 & 3\\ 11 & 2\end{bmatrix}$$

a. $c\left(A+C\right)$

b. cdB

c. $cA-\left(B+C\right)$

THEOREM 2.2: PROPERTIES OF ZERO MATRICES

If A is an $m \times n$ matrix, and c is a scalar, then the following properties are true.

1. $A + O_{mn} =$ _________
2. $A + (-A) =$ ________
3. If $cA = O$, then ___________________________________.

Example 2: Solve for X in the equation, given

$$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}$$

a. $X = 3A - 2B$

b. $2A + 4B = -2X$

THEOREM 2.3: PROPERTIES OF MATRIX MULTIPLICATION

If A, B, and C are matrices (**with sizes such that the given matrix products are defined**), and c is a scalar, then the following properties are true.

1. $A(BC)=$____________________ Associative property of multiplication
2. $A(B+C)=$__________________ Distributive property of multiplication
3. $(A+B)C=$__________________ Distributive property of multiplication
4. $c(AB)=(cA)B=$______________

Example 3: Show that $AC=BC$, even though $A\neq B$.

$$A=\begin{bmatrix}1 & 2 & 3\\ 0 & 5 & 4\\ 3 & -2 & 1\end{bmatrix} \qquad B=\begin{bmatrix}4 & -6 & 3\\ 5 & 4 & 4\\ -1 & 0 & 1\end{bmatrix} \qquad C=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 4 & -2 & 1\end{bmatrix}$$

Example 4: Show that $AB = \mathbf{0}$, even though $A \neq \mathbf{0}$ and $B \neq \mathbf{0}$.

$$A = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

THEOREM 2.4: PROPERTIES OF THE IDENTITY MATRIX

If A is an $m \times n$ matrix, then the following properties are true.

1. $AI_n =$ ______
2. $I_m A =$ ______

THEOREM 2.5: NUMBER OF SOLUTIONS OF A LINEAR SYSTEM

For a system of linear equations, precisely one of the following is true.

1. The system has exactly ________ solution.
2. The system has ____________________ many solutions.
3. The system has ________ solution.

THE TRANSPOSE OF A MATRIX

The transpose of a matrix is denoted ________ and is formed by writing its ________________ as _________.

Example 5: Find the transpose of the matrix.

a. $A=\begin{bmatrix}1 & -1\\ 2 & 9\\ 4 & 10\end{bmatrix}$

b. $A=\begin{bmatrix}6 & -7 & 19\\ -7 & 0 & 23\\ 19 & 23 & -32\end{bmatrix}$

THEOREM 2.6: PROPERTIES OF TRANSPOSES

If A and B are matrices (with sizes such that the given matrix operations are defined), and c is a scalar, then the following properties are true.

1. $\left(A^T\right)^T =$ ____________________ Transpose of a transpose

Proof:

2. $\left(A+B\right)^T =$ __________________ Transpose of a sum

Proof:

3. $\left(cA\right)^T =$ ____________________ Transpose of a scalar multiple

4. $\left(AB\right)^T =$ ____________________ Transpose of a product

Example 6: Find a) $A^T A$ and b) AA^T. Show that each of these products is symmetric.

$$A = \begin{bmatrix} 4 & -3 & 2 & 0 \\ 2 & 0 & 11 & -1 \\ -1 & -2 & 0 & 3 \\ 14 & -2 & 12 & -9 \\ 6 & 8 & -5 & 4 \end{bmatrix}$$

Example 7: A square matrix is called skew-symmetric when $A^T = -A$. Prove that if A and B are skew-symmetric matrices, then $A+B$ is skew-symmetric.

STOCHASTIC MATRICES

Many types of applications involve a finite set of ____________________ ________________________ of a given population. The ____________________ that a member of a population will change from the ___________ state to the ___________ state is represented by a number __________, where __________________. A probability of _____________ means that the member is certain ___________ to change from the *j*th state to the *i*th state whereas a probability of _____________ means that the member is ___________ to change from the *j*th state to the *i*th state.

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix}$$

***P* is called the _____________ of _______________ probabilities.** At each transition, each member in a given state must either stay in that state or change to another state. Therefore, the sum of the entries in any _______________ is _____. This type of matrix is called __________________. An __________ matrix *P* is a **stochastic matrix** when each entry is a number between ______ and ______ inclusive.

Example 8: Determine whether the matrix is stochastic.

$$A = \begin{bmatrix} 0.35 & 0.2 \\ 0.65 & 0.75 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1}{8} & \frac{3}{5} & \frac{1}{12} \\ \frac{1}{2} & \frac{1}{10} & \frac{1}{3} \\ \frac{3}{8} & \frac{3}{10} & \frac{7}{12} \end{bmatrix}$$

Example 9: A medical researcher is studying the spread of a virus in a population of 1000 laboratory mice. During any week, there is an 80% probability that an infected mouse will overcome the virus, and during the same week, there is a 10% probability that a noninfected will become infected. One hundred mice are currently infected with the virus. How many will be infected (a) next week and (b) in two weeks?

Example 10: It has been claimed that the best predictor of today's weather is yesterday's weather. Suppose that in San Diego, if it rained yesterday, then there is a 20% chance of rain today, and if it did not rain yesterday, then there is a 90% chance of no rain today.

a. Find the transition matrix describing the rain probabilities.

b. If it rained Sunday, what is the chance of rain on Tuesday?

c. If it did not rain on Wednesday, what is the chance of rain on Saturday?

d. If the probability of rain today is 30%, what is the chance of rain tomorrow?

2.3: The Inverse of a Matrix

Learning Objectives

1. Find the inverse of a matrix (if it exists)
2. Use properties of inverse matrices
3. Use an inverse matrix to solve a system of linear equations
4. Encode and decode messages
5. Elementary Matrices
6. *LU*-Factorization

DEFINITION OF THE INVERSE OF A MATRIX

An $n \times n$ matrix A is ________________ or ______________________ when there exists an $n \times n$ matrix B such that

where I_n is the ________________ matrix of order n. The matrix B is called the (___________________)

______________ of A. A matrix that does not have an inverse is called noninvertible or _______________.

***Nonsquare matrices do not have _________________.**

Example 1: For the matrices below, show that B is the inverse of A.

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

THEOREM 2.7: UNIQUENESS OF AN INVERSE

If A is an invertible matrix, then its inverse is unique. The inverse of ____ is denoted ______.

Proof:

FINDING THE INVERSE OF A MATRIX BY GAUSS-JORDAN ELIMINATION

Let A be a square matrix of order n.

1. Write the ________________ matrix that consists of the given matrix A on the left and the $n \times n$ ________________ matrix ______ on the right to obtain ___________. This process is called __________________ matrix I to matrix A.
2. If possible, row reduce ________ to _______ using elementary row operations on the entire matrix ___________. The result will be the matrix ____________. If this is not possible, then A is noninvertible (or __________________).
3. Check your work by multiplying to see that _____________________________.

Example 2: Find the inverse of the matrix (if it exists), by solving the matrix equation $AX = I$.

$$A = \begin{bmatrix} 12 & 3 \\ 5 & -2 \end{bmatrix}$$

Example 3: Find the inverse of the matrix (if it exists).

a. $A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

b. $A = \begin{bmatrix} 10 & 5 & -7 \\ -5 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix}$

THEOREM 2.8: PROPERTIES OF INVERSE MATRICES

If A is an invertible matrix, k is a positive integer, and c is a nonzero scalar, then A^{-1}, A^k, cA, and A^T are invertible and the following are true.

1. $\left(A^{-1}\right)^{-1}$ ________

Proof:

2. $\left(A^k\right)^{-1}$ ____

3. $(cA)^{-1}$ ________

Proof:

4. $\left(A^T\right)^{-1}$ ________

THEOREM 2.9: THE INVERSE OF A PRODUCT

If A and B are invertible matrices of order n, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof:

Example 4: Use the inverse matrices below for the following problems.

$$A^{-1} = \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & -\frac{1}{11} \end{bmatrix}$$

a. $(AB)^{-1}$

b. $(A^T)^{-1}$

c. $(7A)^{-1}$

THEOREM 2.10: CANCELLATION PROPERTIES

If C is an **invertible matrix**, then the following properties hold true.

1. If $AC = BC$ then $A = B$. Right cancellation property

Proof:

2. If $CA = CB$ then $A = B$. Left cancellation property

THEOREM 2.11: SYSTEMS OF EQUATIONS WITH UNIQUE SOLUTIONS

If A is an invertible matrix, then the system of linear equations $A\mathbf{x}=\mathbf{b}$ has a unique solution given by $\mathbf{x}=A^{-1}\mathbf{b}$.

Proof:

CRYPTOGRAPHY

A ________________________ is a message written according to a secret code. Suppose we assign a number to each letter in the alphabet.

0	_	**14**	**N**
1	A	15	O
2	B	16	P
3	C	17	Q
4	D	18	R
5	E	19	S
6	F	20	T
7	G	21	U
8	H	22	V
9	I	23	W
10	J	24	X
11	K	25	Y
12	L	26	Z
13	M		

Example 5: Write the uncoded row matrices of size 1 x 3 for the message TARGET IS HOME.

Example 6: Use the following invertible matrix to encode the message TARGET IS HOME.

$$A = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

Example 7: How would you decode a message?

DEFINITION OF AN ELEMENTARY MATRIX

An $n \times n$ matrix is called an _________________ matrix when it can be obtained from the

____________________ matrix ______ by a single elementary _________ operation.

Example 8: Identify the matrices that are elementary below.

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -1 & -3 \end{bmatrix}$$

THEOREM 2.12: REPRESENTING ELEMENTARY ROW OPERATIONS

Let E be the ____________________ matrix obtained by performing an elementary row operation on_______.

If that same elementary row operation is performed on an _____________ matrix A, then the resulting

matrix is given by the product ________.

Example 9: Given A and C below

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 4 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}$$

find an elementary matrix E such that $EA = C$.

Example 10: Find a sequence of elementary matrices that can be used to write the matrix in row-echelon form.

Equivalent matrix to A | Elementary Row Op, | Elementary Matrix

$$A = \begin{bmatrix} 0 & 3 & -3 & 6 \\ 1 & -1 & 2 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

DEFINITION OF ROW EQUIVALENCE

Let A and B be $m \times n$ matrices. Matrix B is ____________________________ to A when there exists a finite number of ______________________ matrices, ______________________________ such that

THEOREM 2.13: ELEMENTARY MATRICES ARE INVERTIBLE

If E is an elementary matrix, then E^{-1} exists and is an ____________________________ matrix.

Example 11: Find the inverse of the elementary matrix.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

THEOREM 2.14: EQUIVALENT CONDITIONS

If A is an $n \times n$ matrix, then the following statements are equivalent.

1. A is ______________________.
2. $A\mathbf{x} = \mathbf{b}$ has a ___________________ solution for every __________ column matrix ______.
3. $A\mathbf{x} = \mathbf{0}$ has only the ____________________ solution.
4. A is _____________________ to ______.
5. A can be written as the product of _____________________ matrices.

THE *LU*-FACTORIZATION

DEFINITION OF *LU*-FACTORIZATION

If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then $A = LU$ is an ***LU*-factorization** of A.

Example 12: Solve the linear system $A\mathbf{x} = \mathbf{b}$ by

1. Finding an *LU*-factorization of the coefficient matrix A.
2. Solving the lower triangular system $L\mathbf{y} = \mathbf{b}$.
3. Solving the upper triangular system $U\mathbf{x} = \mathbf{y}$.

$$\begin{aligned}
2x_1 \qquad\qquad\qquad &= 4 \\
-2x_1 + x_2 - x_3 \qquad &= -4 \\
6x_1 + 2x_2 + x_3 \qquad &= 15 \\
- x_4 &= -1
\end{aligned}$$

2.5: Linear Transformations

Learning Objectives

1. Find the preimage and image of a function
2. Determine if a function is a linear transformationWrite and use a stochastic matrix

IMAGES AND PREIMAGES OF FUNCTIONS

In this section we will learn about functions that _____________ a vector space _____ onto a vector space _____. This is denoted by ____________. The standard function terminology is used for such functions. ____ is called the __________________ of _____, and _____ is called the ____________ of _____. If $\mathbf{v}$ is in V, and $\mathbf{w}$ in W such that _______________, _____ is called the ________________ of ____ under ______. The set of all images of vectors in V is called the _______________ of ______, and the set of all $\mathbf{v}$ in V such that __________________ is called the _____________________ of _______.

Example 1: Use the function to find (a) the image of $\mathbf{v}$ and (b) the preimage of $\mathbf{w}$.

$T(v_1, v_2) = (2v_2 - v_1, v_1, v_2)$, $\mathbf{v} = (0,6)$, $\mathbf{w} = (3,1,2)$

DEFINITION OF A LINEAR TRANSFORMATION

Let V and W be vector spaces. The function $T: V \to W$ is called a linear transformation of _____ into _____ when the following two properties are true for all $\mathbf{u}$ and $\mathbf{v}$ in V and any scalar c.

1. ___________________________
2. ___________________________

A linear transformation is _______________ ____________________ because the same result occurs whether you perform the operations of addition and scalar multiplication ________________ or ________________ applying the _________________ _______________________. Although the same symbols denote the vector operations in both V and W, you should note that the operations may be different.

Example 2: Determine whether the function is a linear transformation.

a. $T: R^3 \to R^3,\ T(x, y, z) = (x+1, y+1, z+1)$

b. $T: M_{2,2} \to R,\ T(A) = a + b + c + d$

THEOREM 2.15: PROPERTIES OF LINEAR TRANSFORMATIONS

Let T be a linear transformation from V into W, where $\mathbf{u}$ and $\mathbf{v}$ are in V. Then the following properties are true.

1. ____________________________

2. ____________________________

3. ____________________________

Proof:

4. If ____,

   then _____________________________________ = ____

Example 3: Let $T : R^3 \to R^3$ be a linear transformation such that $T(1,0,0) = (2,4,-1)$, $T(0,1,0) = (1,3,-2)$, and $T(0,0,1) = (0,-2,2)$. Find the indicated image.

$T(2,-1,0)$

THEOREM 2.16: THE LINEAR TRANSFORMATION GIVEN BY A MATRIX

Let A be an $m \times n$ matrix. The function T defined by

is a linear transformation from R^n into R^m. In order to conform to matrix multiplication with an $m \times n$ matrix, $n \times 1$ matrices represent the vectors in R^n and $m \times 1$ matrices represent the vectors in R^m.

$$A\mathbf{v} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + & \cdots & +a_{1n}v_n \\ \vdots & \ddots & \vdots \\ a_{m1}v_1 + & \cdots & +a_{mn}v_n \end{bmatrix}$$

Example 4: Define the linear transformation $T : R^n \to R^m$ by $T(\mathbf{v}) = A\mathbf{v}$. Find the dimensions of R^n and R^m.

a. $A = \begin{bmatrix} 1 & 2 \\ -2 & 4 \\ -2 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 2 & 1 & -4 & 1 \end{bmatrix}$

Example 5: Consider the linear transformation from Example 4, part a.

a. Find $T(2,4)$

b. Find the preimage of $(-1, 2, 2)$

c. Explain why the vector $(1,1,1)$ has no preimage under this transformation.

PART 2: DETERMINANTS, GENERAL VECTOR SPACES, AND MATRIX REPRESENTATIONS OF LINEAR TRANSFORMATIONS

3.1: THE DETERMINANT OF A MATRIX

Learning Objectives

1. Find the determinant of a 2 x 2 matrix
2. Find the minors and cofactors of a matrix
3. Use expansion by cofactors to find the determinant of a matrix
4. Find the determinant of a triangular matrix
5. Use elementary row operations to evaluate a determinant
6. Use elementary column operations to evaluate a determinant
7. Recognize conditions that yield zero determinants

Every ______________ matrix can be associated with a real number called its _____________________.

Historically, the use of determinants arose from the recognition of special ___________________ that occur in the __________________ of systems of linear equations.

DEFINITION OF THE DETERMINANT OF A 2 x 2 MATRIX

The _______________________ of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by $\det(A) =$ _____________________________.

**Note: In this text, ______________ and ___________ are used interchangeably to represent the determinant of a matrix. In this context, the vertical bars are used to represent the __________________ of a matrix as opposed to the __________________ value.

Example 1:

a. Find $\det(A)$ and $\det(B)$.

$$A = \begin{bmatrix} -1 & 4 \\ 11 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 21 & -3 \\ -6 & 10 \end{bmatrix}$$

Check this out…

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

b. Find A^{-1} and B^{-1}

$A = \begin{bmatrix} -1 & 4 \\ 11 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 21 & -3 \\ -6 & 10 \end{bmatrix}$

DEFINITION OF MINORS AND COFACTORS OF A MATRIX

If A is a _______________ matrix, then the _______________ _______ of the element _____ is the determinant of the matrix obtained by deleting the ____ row and the _____ column of A. The ________________________ _____ is given by C_{ij} ______________________________.

Example 2: Find the minor and cofactor of a_{12} and b_{13}.

a. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

b. $B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$

DEFINITION OF THE DETERMINANT OF A SQUARE MATRIX

If A is a _______________ matrix of order $n > 2$, then the ___________________ of A is the ______ of the entries in the first row of A multiplied by their respective ______________________. That is,

$$\det(A) = |A| = \sum_{j=1}^{n} a_{1j}C_{1j} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

Example 3: Confirm that, for 2x2 matrices, this definition yields $|A| = a_{11}a_{22} - a_{21}a_{12}$.

Example 4: Find $|B|$.

$$B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

THEOREM 3.1: EXPANSION BY COFACTORS

If A be a square matrix of order n. Then the determinant of A is given by

$$\det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \text{ (}i\text{th row expansion)}$$

$$\det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \text{ (}j\text{th column expansion)}$$

Is there an easier way to complete the previous example?

$$B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

Alternative Method to evaluate the determinant of a 3 x 3 matrix: Copy the first and second columns of the matrix to form fourth and fifth columns. Then obtain the determinant by adding (or subtracting) the products of the six diagonals.

$$B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

Example 5: Find $\det(A)$ and $\det(B)$.

$$A = \begin{bmatrix} 1 & 0 & 2 & 6 \\ 3 & 7 & -1 & 0 \\ 6 & -1 & 2 & 5 \\ -3 & 5 & -8 & 7 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 0 & 0 \\ 1 & 1 & 0 \\ 7 & -2 & 11 \end{bmatrix}$$

What did you notice?

THEOREM 3.2: DETERMINANT OF A TRIANGULAR MATRIX

If A is a triangular matrix of order n, then its determinant is the _________________ of the ____________ on the ____________ ___________________. That is, $\det(A)=|A|=$ __________________________________.

Example 6: Find the values of λ, for which the determinant is zero.

$$\begin{vmatrix} \lambda-1 & 1 \\ 4 & \lambda-3 \end{vmatrix}$$

Consider the following matrix:

$$A=\begin{bmatrix} -1 & 2 & 1 \\ 3 & 4 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

Find the determinant.

Now let's put the matrix into row-echelon form. In other words, row reduce to an upper triangular matrix. Keep track of each elementary row operation.

What's the determinant of this matrix?

Take a closer look at the determinants of the two matrices. Do you notice anything?

THEOREM 3.3: ELEMENTARY ROW OPERATIONS AND DETERMINANTS

Let A and B be square matrices.

1. When B is obtained from A by ____________________ (_______________) two ________ of A,

   ______________________.

2. When B is obtained from A by _____________ a ____________________ of a row of A to another row of A, __________________. To clarify, the "new" row is not scaled, but the row used to get the new row can be scaled. If the new row is scaled, you also use #3 below.

3. When B is obtained from A by ________________________ a row of A by a _______________

   _________________ c, ________________________.

NOTE: Theorem 3.3 remains valid when the word "column" replaces the word "row". Operations performed on columns are called elementary column operations.

Example 7: Determine which property of determinants the equation illustrates.

a. $\begin{vmatrix} 1 & -1 & 3 \\ 4 & 12 & 7 \\ 3 & -3 & 8 \end{vmatrix} = -\begin{vmatrix} 3 & -1 & 1 \\ 7 & 12 & 4 \\ 8 & -3 & 3 \end{vmatrix}$

b. $\begin{vmatrix} 2 & -4 & 2 \\ 6 & 10 & 2 \\ 8 & -4 & 6 \end{vmatrix} = 8\begin{vmatrix} 1 & -2 & 1 \\ 3 & 5 & 1 \\ 4 & -2 & 3 \end{vmatrix}$

Example 8: Use elementary row or column operations to find the determinant of the matrix.

$$A = \begin{bmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 4 & 1 & 6 \end{bmatrix}$$

THEOREM 3.4: CONDITIONS THAT YIELD A ZERO DETERMINANT

If A is a square matrix, and any one of the following conditions is true, then $\det(A) = 0$.

1. An entire ________ (or ________________) consists of ___________.
2. Two __________ (or ________________) are _______________.
3. One ________ (or ______________) is a _______________ of another _______ (or______________).

	Cofactor Expansion		Row Reduction	
Order n	Additions	Multiplications	Additions	Multiplications
3	5	9	5	10
5	119	205	30	45
10	3,628,799	6,235,300	285	339

Example 9: Prove the property.

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),\ a \neq 0,\ \mathrm{b} \neq 0,\ \mathrm{c} \neq 0.$$

3.2: PROPERTIES OF DETERMINANTS

Learning Objectives

1. Find the determinant of a matrix product and a scalar multiple of a matrix
2. Find the determinant of an inverse matrix and recognize equivalent conditions for a nonsingular matrix
3. Find the determinant of the transpose of a matrix
4. Use Cramer's Rule to solve a system of linear equations
5. Use determinants to find area, volume, and equations of lines and planes

Example 1: Find $|A|, |B|, |A||B|, |A+B|, |A|+|B|$ and $|AB|$.

$$A=\begin{bmatrix}3 & 2 & 1\\ 1 & -1 & 2\\ 3 & 1 & 0\end{bmatrix} \qquad B=\begin{bmatrix}2 & -1 & 4\\ 0 & 1 & 3\\ 3 & -2 & 1\end{bmatrix}$$

THEOREM 3.5: DETERMINANT OF A MATRIX PRODUCT

If A and B are square matrices of order n, then

Example 2: Find $|3A|$ and $|3B|$.

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

THEOREM 3.6: DETERMINANT OF A SCALAR MULTIPLE OF A MATRIX

If A is a square matrix of order n and c is a scalar, then the determinant of $|cA|$ is

Proof:

Example 3: Find A^{-1}, $|A|$, $|A^{-1}|$, B^{-1}, $|B^{-1}|$, and $|B|$.

$A=\begin{bmatrix}-3 & 6\\ -2 & 4\end{bmatrix}$

$B=\begin{bmatrix}5 & 2\\ 11 & 7\end{bmatrix}$

THEOREM 3.7: DETERMINANT OF AN INVERTIBLE MATRIX

A square matrix A is invertible (nonsingular) if and only if

Example 4: Find $|A|$ and $|A^{-1}|$.

$$A=\begin{bmatrix}-3 & 3\\ -2 & 1\end{bmatrix}$$

THEOREM 3.8: DETERMINANT OF AN INVERSE MATRIX

If A is an $n \times n$ invertible matrix, then

Proof:

EQUIVALENT CONDITIONS FOR A NONSINGULAR MATRIX

If A is an $n \times n$ matrix, then the following statements are equivalent.

1. A is _______________________.
2. $A\mathbf{x} = \mathbf{b}$ has a _________________ solution for every ___________ column matrix.
3. $A\mathbf{x} = \mathbf{0}$ has only the _______________ solution.
4. A is _____________________________ to _________.
5. A can be written as the product of __________________________ matrices.
6. ________________________.

Example 5: Determine if the system of linear equations has a unique solution.

$$\begin{aligned} x_1 + x_2 - x_3 &= 4 \\ 2x_1 - x_2 - x_3 &= 6 \\ 3x_1 - 2x_2 + 2x_3 &= 0 \end{aligned}$$

Example 6: Find $|A|$ and $|A^T|$.

$$A = \begin{bmatrix} 7 & 12 \\ 2 & -2 \end{bmatrix}$$

THEOREM 3.9: DETERMINANT OF A TRANSPOSE

If A is a square matrix, then

Example 7: Solve the system of linear equations. Assume that $a_{11}a_{22} - a_{21}a_{12} \neq 0$.

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

THEOREM 3.10: CRAMER'S RULE

If a system of n linear equations in n variables has a coefficient matrix A with a nonzero determinant $|A|$, then the solution of the system is

Where the *j*th column of A_j is the column of constants in the system of equations.

Example 8: If possible, use Cramer's Rule to solve the system.

a.

$$\begin{aligned} -x_1 - 2x_2 &= 7 \\ 2x_1 + 4x_2 &= 11 \end{aligned}$$

b.

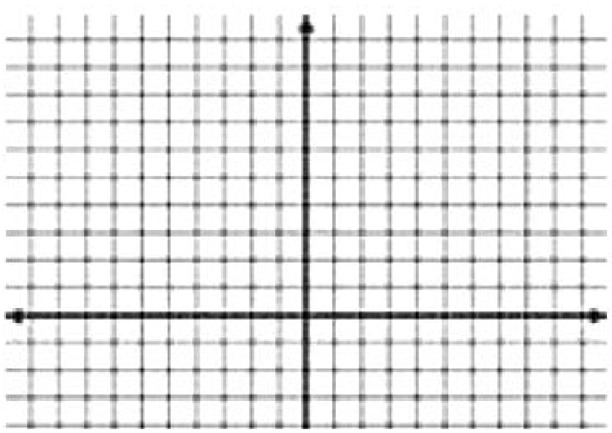
$$\begin{aligned} -8x_1 + 7x_2 - 10x_3 &= -151 \\ 12x_1 + 3x_2 - 5x_3 &= 86 \\ 15x_1 - 9x_2 + 2x_3 &= 187 \end{aligned}$$

AREA OF A TRIANGLE IN THE xy-PLANE

The area of a triangle with vertices $(x_1, y_1),\ (x_2, y_2),$ and (x_3, y_3) is

where the sign ($\pm$) is chosen to give positive area.

Proof:



Example 9: Find the area of the triangle whose vertices are $(1,-1)$, $(3,-5)$, and $(0,-2)$.

TEST FOR COLLINEAR POINTS IN THE *xy*-PLANE

Three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear if and only if

TWO-POINT FORM OF THE EQUATION OF A LINE

An equation of the line passing through the distinct points (x_1, y_1) and (x_2, y_2) is given by

VOLUME OF A TETRAHEDRON

The volume of a tetrahedron with vertices (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) is

where the sign ($\pm$) is chosen to give positive volume.

Example 11: Find the volume of the tetrahedron with vertices $(1,1,1)$, $(0,0,0)$, $(2,1,-1)$, and $(-1,1,2)$.

TEST FOR COPLANAR POINTS IN SPACE

Four points, (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) are coplanar if and only if

THREE-POINT FORM OF THE EQUATION OF A LINE

An equation of the plane passing through the distinct points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is given by

3.3: GENERAL VECTOR SPACES

Learning Objectives:

1. Determine whether a set of vectors is a vector space
2. Determine if a subset of a known vector space V is a subspace of V
3. Write a vector as a linear combination of other vectors
4. Recognize bases in the vector spaces R^n, P_n, and $M_{m,n}$
5. Determine whether a set S of vectors in a vector space V is a basis for V
6. Find the dimension of a vector space

DEFINITION OF A VECTOR SPACE

Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the listed axioms are satisfied for every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and every scalar (real number) c and d, then V is called a **vector space**.

Addition

1. $\mathbf{u}+\mathbf{v}$ is in V. ____________ under addition
2. $\mathbf{u}+\mathbf{v}=$ _____________________ _______________ property
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=$ ______________ _______________ property
4. V has a ____________ vector ______ such that additive ________________

   for every _____ in V, _____________.

5. For every ____ in V, there is a vector in V additive ________________

   denoted by ____ such that ________________.

Scalar Multiplication

6. $c\mathbf{u}$ is in____. _________ under scalar mult.
7. $c(\mathbf{u}+\mathbf{v})=$ ______________ _________________ property
8. $(c+d)\mathbf{u}=$ _____________ _________________ property
9. $c(d\mathbf{u})=$ ________ _________________ property
10. $1(\mathbf{u})=$ ____ _________________identity

THEOREM 3.11: PROPERTIES OF SCALAR MULTIPLICATION

Let $\mathbf{v}$ be any element of a vector space V, and let c be any scalar. Then the following properties are true.

1. $0\mathbf{v} =$ __________
2. $c\mathbf{0} =$ __________
3. If __________, then __________ or _________.
4. $(-1)\mathbf{v} =$ ________

Example 1: Determine whether the set, together with the indicated operations, is a vector space. If it is not, then identify at least one of the ten vector space axioms that fails.

a. The set of all 2 x 2 matrices of the form $S = \left\{ \begin{bmatrix} a & b \\ c & 1 \end{bmatrix} : a, b, c, d \in R \right\}$.

b. The set of all 2 x 2 nonsingular matrices with the standard operations.

IMPORTANT VECTOR SPACES CONTINUED

_____________ = the set of all ________________ ______________ defined on the real ________________ line.

_____________ = the set of all ________________ ______________ defined on a ______________

_________________ ___________.

_____ = the set of all _______________________.

_____ = the set of all _______________________ of degree ________.

_________ = the set of all _____________ matrices.

_________ = the set of all _____________ _____________ matrices.

Example 2: Describe the zero vector (the additive identity) of the vector space.

a. $C(-\infty, \infty)$

b. $M_{1,4}$

Example 3: Describe the additive inverse of a vector in the vector space.

a. $C(-\infty, \infty)$

b. $M_{1,4}$

Example 4: Determine whether the set of continuous functions, $C(-\infty, \infty)$ is a vector space.

1. Closure under addition.

2. Commutativity under addition.

3. Associativity under addition.

4. Additive identity.

5. Additive inverse.

6. Closure under scalar multiplication.

7. Distributivity under scalar multiplication (2 vectors and 1 scalar).

8. Distributivity under scalar multiplication (2 scalars and 1 vector).

9. Associativity under scalar multiplication.

10. Scalar multiplicative identity.

Conclusion?

Example 5: Determine whether the set W is a subspace of the vector space V with the standard operations of addition and scalar multiplication.

a. $V : C[-1,1]$

   W : The set of all functions that are differentiable on $[-1,1]$

b. $V: C(-\infty, \infty)$

W: The set of all negative functions: $f(x) < 0$.

c. $V: C(-\infty, \infty)$

W: The set of all odd functions: $f(-x) = -f(x)$.

d. $V:\left\{M_{n,n}:n\in Z^{+}\right\}$

W : The set of all n x n diagonal matrices.

e. W : The set of all n x n matrices whose trace is nonzero.

$V:\left\{M_{n,n}:n\in Z^{+}\right\}$

f. $V:C\left(-\infty,\infty\right)$

$W:\left\{ax+b:a,b\in R,a\neq 0\right\}$

g. $V:\{M_{m,n} : m,n \in Z^+\}$

$W:\left\{\begin{bmatrix} a & 0 & \sqrt{a} \end{bmatrix}^T : a \in R, a \geq 0\right\}$

Example 6: For the matrices

$A=\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix}$

in $M_{2,2}$, determine whether the given matrix is a linear combination of A and B.

$\begin{bmatrix} 6 & -19 \\ 10 & 7 \end{bmatrix}$

Consider $P_n(x) =$ ____

Example 7: Determine whether the set of vectors in P_2 is linearly independent or linearly dependent.

$S = \left\{x^2, x^2 + 1\right\}$

Example 8: Determine whether the set of vectors in $M_{2,2}$ is linearly independent or linearly dependent.

$$S = \left\{\begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} -8 & -3 \\ -6 & 17 \end{bmatrix}\right\}$$

Example 9: Write the standard basis for the vector space.

a. $M_{3,2}$

b. P_3

Example 10: Determine whether S is a basis for the indicated vector space.

$S = \left\{4t - t^2, 5 + t^3, 3t + 5, 2t^3 - 3t^2\right\}$ for P_3

Example 11: Find a basis for the vector space of all 3 x 3 symmetric matrices. What is the dimension of this vector space?

Example 11: Let T be the linear transformation from P_2 into R given by the integral $T(p)=\int_0^1 p(x)dx$. Find the preimage of 1. That is, find the polynomial function(s) of degree 2 or less such that $T(p)=1$.

3.4: RANK/NULLITY OF A MATRIX, SYSTEMS OF LINEAR EQUATIONS. AND COORDINATE VECTORS

Learning Objectives:

1. Find a basis for the row space, a basis for the column space, and the rank of a matrix
2. Find the nullspace of a matrix
3. Find a coordinate matrix relative to a basis in R^n
4. Find the transition matrix from the basis B to the basis B' in R^n
5. Represent coordinates in general *n*-dimensional spaces

Let's do our math stretches!

Consider the following matrix.

$$A = \begin{bmatrix} 1 & 3 & -1 & 5 \\ 7 & 1 & 13 & 6 \end{bmatrix}$$

The row vectors of A are:

The column vectors of A are:

DEFINITION OF ROW SPACE AND COLUMN SPACE OF A MATRIX

Let A be an $m \times n$ matrix.

The _______ space of A is the ____________ of R^n _______________ by the ________ vectors of A.

The ______________ space of A is the subspace of R^n ______________ by the _______________ vectors of A.

Recall that two matrices are row-equivalent when one can be obtained from the other by ____________________ _______ operations.

THEOREM 3.12: ROW-EQUIVALENT MATRICES HAVE THE SAME ROW SPACE

If an $m \times n$ matrix A is row-equivalent to an $m \times n$ matrix B, then the row space of A is equal to the row space of B.

Proof:

THEOREM 3.12: BASIS FOR THE ROW SPACE OF A MATRIX

If a matrix A is row-equivalent to a matrix B in row-echelon form, then the nonzero row vectors of B form a ____________ for the row space of A.

To find a basis for the row space of a matrix: _________ reduce the matrix. The __________________ rows in the ____________________ matrix are a ________________ for the row space of the matrix. Your answer should be in the form of a _________ of ________ vectors.

To find a basis for the column space of a matrix:

Method 1: Use the steps above on the transpose of the matrix. Your answer should be in the form of a ______ of _________________ vectors.

Method 2: Use reduced form of the original matrix to find the columns which contain the ______________ (leading __________). Use the corresponding columns from the __________________ matrix for a basis. Your answer should be in the form of a ______ of _________________ vectors.

Example 1: Find a basis for the row space and column space of the following matrix:

$$A=\begin{bmatrix} 2 & -3 & 1 \\ 5 & 10 & 6 \\ 8 & -7 & 5 \end{bmatrix}$$

Example 2: Find a basis for the row space and column space of the following matrix:

$$A=\begin{bmatrix} 4 & 20 & 31 \\ 6 & -5 & -6 \\ 2 & -11 & -16 \end{bmatrix}$$

THEOREM 3.13: ROW AND COLUMN SPACES HAVE EQUAL DIMENSIONS

If A is an $m \times n$ matrix, then the row space and the column space of A have the same ___________________.

DEFINITION OF THE RANK OF A MATRIX

The ___________________ of the ________ (or _________________) space of a matrix A is called the

__________ of A and is denoted by ________________.

Example 3: Find the rank of the matrix from

a. Example 1

b. Example 2

THEOREM 3.14: SOLUTIONS OF A HOMOGENEOUS SYSTEM

If A is an $m \times n$ matrix, then the set of all solutions of the homogeneous system of linear equations

_______________ is a _________________ of _____ called the ___________________ of _____ and is denoted

____________. So,

The ____________________ of the nullspace of A is called the ____________ of ____.

Proof:

Example 4: Find the nullspace of the following matrix A, and determine the nullity of A.

$$A = \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ -2 & -8 & -4 & -2 \end{bmatrix}$$

THEOREM 3.15: DIMENSION OF THE SOLUTION SPACE

If A is an $m \times n$ matrix of rank _____, then the ___________________ of the solution space of

_______________ is _____________. That is, Example 5: consider the following homogeneous system of linear equations:

$$\begin{aligned} x - y &= 0 \\ -x + y &= 0 \end{aligned}$$

a. Find a basis for the solution space.

b. Find the dimension of the solution space.

c. Find the solution of a consistent system $A\mathbf{x} = \mathbf{b}$ in the form $\mathbf{x}_p + \mathbf{x}_h$

THEOREM 3.16: SOLUTIONS OF A NONHOMOGENEOUS LINEAR SYSTEM

If $\mathbf{x}_p$ is a particular solution of the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$, then every solution of this system can be written in the form _________________ where $\mathbf{x}_h$ is a solution of the corresponding homogeneous system___________.

Proof:

THEOREM 3.17: SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS

The system ____________ is consistent if and only if ____ is in the column space of ______.

Proof:

Example 7: consider the following nonhomogeneous system of linear equations:

$$\begin{aligned} 2x - 4y + 5z &= 8 \\ -7x + 14y + 4z &= -28 \\ 3x - 6y + z &= 12 \end{aligned}$$

Determine whether $A\mathbf{x} = \mathbf{b}$ is consistent.

If the system is consistent, write the solution in the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_p$ is a particular solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_h$ is a solution of $A\mathbf{x} = \mathbf{0}$.

COORDINATE REPRESENTATION RELATIVE TO A BASIS

Let $B=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ be an ordered basis for a vector space V , and let $\mathbf{x}$ be a vector in V such that

The scalars $c_1,c_2,\ldots,c_n$ are called the ________________ of ____ relative to the __________ _____. The

______________ matrix (or coordinate __________) of _____ relative to ____ is the ____________ matrix in

_____ whose ___________________ are the coordinates of _____.

Note: In _____, column notation is used for the coordinate matrix. For the vector _______________________,

the ______ are the coordinates of ____ (relative to the _______________ ______________ ____ for _____. So

you have

Example 8: Find the coordinate matrix of $\mathbf{x}$ in R^n relative to the standard basis.

$\mathbf{x}=(1,-3,0)$

Example 9: Given the coordinate matrix of $\mathbf{x}$ relative to a (nonstandard) basis B for R^n, find the coordinate matrix of $\mathbf{x}$ relative to the standard basis.

$B = \{(4,0,7,3),(0,5,-1,-1),(-3,4,2,1),(0,1,5,0)\}$

$$[\mathbf{x}]_B = \begin{bmatrix} -2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$

Example 10: Find coordinate matrix of $\mathbf{x}$ in R^n relative to the basis B'.

$B' = \{(-6,7),(4,-3)\}$, $\mathbf{x} = (-26,32)$

The matrix _____ is called the _______________ ____________ from ____ to _____, where ________ is the coordinate matrix of ____ relative to _____, and ______ is the coordinate matrix of _____ relative to _____. Multiplication by the transition matrix ____ changes a coordinate matrix relative to _____ into a coordinate matrix relative to _____.

Change of basis from _____ to ____:

Change of basis from _____ to ____:

The change of basis problem in example 10 can be represented by the matrix equation:

THEOREM 3.18: THE INVERSE OF A TRANSITION MATRIX

If P is the transition matrix from a basis B' to a basis B in R^n, then ____ is invertible and the transition matrix from ____ to ____ is given by _____.

LEMMA

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be two bases for a vector space V. If

$$\begin{aligned}
\mathbf{v}_1 &= c_{11}\mathbf{u}_1 + c_{21}\mathbf{u}_2 + \cdots c_{n1}\mathbf{u}_n \\
\mathbf{v}_2 &= c_{12}\mathbf{u}_1 + c_{22}\mathbf{u}_2 + \cdots c_{n2}\mathbf{u}_n \\
&\vdots \\
\mathbf{v}_n &= c_{1n}\mathbf{u}_1 + c_{2n}\mathbf{u}_2 + \cdots c_{nn}\mathbf{u}_n
\end{aligned}$$

then the transition matrix from ____ to ____ is

$$Q = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

THEOREM 3.19: TRANSITION MATRIX FROM B TO B'

Let $B=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ and $B'=\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n\}$ be two bases for R^n . Then the transition matrix _______ from _____ to _____ can be found using Gauss-Jordan elimination on the $n\times 2n$ matrix $\begin{bmatrix} B' & B \end{bmatrix}$ as follows.

Note: The transition matrix from ______ to _____ can be found using Gauss-Jordan elimination on the

_________ matrix _______________ as follows.

Example 11: Find the transition matrix from B to B'.

$B=\{(1,1),(1,0)\}$, $B'=\{(1,0),(0,1)\}$

Example 12: Find the coordinate matrix of p relative to the standard basis for P_3.

$p=3x^2+114x+13$

3.5: THE KERNEL, RANGE, AND MATRIX REPRESENTATIONS OF LINEAR TRANSFORMATIONS, AND SIMILAR MATRICES

Learning Objectives:

1. Find the kernel of a linear transformation
2. Find a basis for the range, the rank, and the nullity of a linear transformation
3. Determine whether a linear transformation is one-to-one or onto
4. Determine whether vector spaces are isomorphic
5. Find the standard matrix for a linear transformation
6. Find the standard matrix for the composition of linear transformations and find the inverse of an invertible linear transformation
7. Find the matrix for a linear transformation relative to a nonstandard basis
8. Find and use a matrix for a linear transformation
9. Show that two matrices are similar and use the properties of similar matrices

THE KERNEL OF A LINEAR TRANSFORMATION

We know from an earlier theorem that for any linear transformation ______________, the zero vector in ____ maps to the __________ vector in _____. That is, _____________. In this section, we will consider whether there are other vectors ____ such that ______________. The collection of all such __________________ is called the _______________ of _____. Note that the zero vector is denoted by the symbol_____ in both _____ and _____, even though these two zero vectors are often different.

DEFINITION OF KERNEL OF A LINEAR TRANSFORMATION

Let $T:V \to W$ be a linear transformation. Then the set of all vectors $\mathbf{v}$ in V that satisfy _______________ is called the ________________ of T and is denoted by _________________.

Example 1: Find the kernel of the linear transformation.

a. $T:R^3 \to R^3, T(x,y,z)=(x,0,z)$

b. $T: P_3 \to P_2, T\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) = a_1 + 2a_2x + 3a_3x^2$

c.
$T: P_2 \to R,$
$T(p) = \int_0^1 p(x)dx$

THEOREM 3.20: THE KERNEL IS A SUBSPACE OF V

The kernel of a linear transformation $T: V \to W$ is a subspace of the domain V.

Proof:

THEOREM 3.20: COROLLARY

Let $T : R^n \to R^m$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. Then the kernel of T is equal to the solution space of ____________________.

THEOREM 3.21: THE RANGE OF *T* IS A SUBSPACE OF *W*

The range of a linear transformation $T : V \to W$ is a subspace of W.

THEOREM 3.21: COROLLARY

Let $T : R^n \to R^m$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. Then the column space of ________ is equal to the _______________ of _____.

Example 2: Let $T(\mathbf{v}) = A\mathbf{v}$ represent the linear transformation T. Find a basis for the kernel of T and the range of T.

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$$

DEFINITION OF RANK AND NULLITY OF A LINEAR TRANSFORMATION

Let $T:V \to W$ be a linear transformation. The dimension of the kernel of T is called the

____________________ of T and is denoted by ______________________. The dimension of the range of T

is called the ______________ of T and is denoted by ____________________.

THEOREM 3.22: SUM OF RANK AND NULLITY

Let $T:V \to W$ be a linear transformation from an *n*-dimensional vector space V into a vector space W. Then

the ________ of the ______________________ of the _______________ and ______________________ is

equal to the dimension of the ___________________. That is,

Proof:

Example 3: Define the linear transformation T by $T(\mathbf{x}) = A\mathbf{x}$. Find $\ker(T)$, $\text{null}(T)$, $\text{range}(T)$, and $\text{rank}(T)$.

$$A = \begin{bmatrix} 3 & -2 & 6 & -1 & 15 \\ 4 & 3 & 8 & 10 & -14 \\ 2 & -3 & 4 & -4 & 20 \end{bmatrix}$$

Example 4: Let $T: R^3 \to R^3$ be a linear transformation. Use the given information to find the nullity of T and give a geometric description of the kernel and range of T.

T is the reflection through the *yz*-coordinate plane:

$T(x, y, z) = (-x, y, z)$

ONE-TO-ONE AND ONTO LINEAR TRANSFORMATIONS

If the _______________ vector is the only vector ____ such that _____________, then ____ is _______________________. A function _______________ is called one-to-one when the __________________ of every _____ in the range consists of a _________________ vector. This is equivalent to saying that ____ is one-to-one if and only if, for all ____ and ____ in _____, ____________________ implies that _______________.

THEOREM 3.23: ONE-TO-ONE LINEAR TRANSFORMATIONS

Let $T:V \to W$ be a linear transformation. Then T is one-to-one if and only if ________________________.

Proof:

THEOREM 3.24: ONTO LINEAR TRANSFORMATIONS

Let $T: V \to W$ be a linear transformation, where W is finite dimensional. Then T is onto if and only if the ____________ of T is equal to the ______________________ of W.

Proof:

THEOREM 3.25: ONE-TO-ONE AND ONTO LINEAR TRANSFORMATIONS

Let $T: V \to W$ be a linear transformation with vector spaces V and W, ___________ of dimension *n*. Then T is one-to-one if and only if it is ________.

Example 5: Determine whether the linear transformation is one-to-one, onto, or neither.

$T: R^2 \to R^2, T(x, y) = (x - y, y - x)$

DEFINITION: ISOMORPHISM

A linear transformation $T: V \to W$ that is _______________________ and ____________ is called an _________________________. Moreover, if V and W are vector spaces such that there exists an isomorphism from V to W, then V and W are said to be ________________________ to each other.

THEOREM 3.26: ISOMORPHIC SPACES AND DIMENSION

Two finite dimensional vector spaces V and W are ________________________if and only if they are of the same _________________________.

Example 6: Determine a relationship among *m, n, j,* and *k* such that $M_{m,n}$ is isomorphic to $M_{j,k}$.

WHICH FORMAT IS BETTER? WHY?

Consider $T: R^3 \to R^3, T(x_1, x_2, x_3) = (4x_1 - x_2 - 5x_3, -2x_1 + x_2 + 6x_3, x_2 - 3x_3)$
and

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 4 & -1 & -5 \\ -2 & 1 & 6 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

What do you think?

The key to representing a linear transformation ______________ by a matrix is to determine how it acts on a _______________ for _____. Once you know the _______________ of every vector in the ______________, you can use the properties of linear transformations to determine _______ for any ___ in ____.

Do you remember the standard basis for R^n? Write this standard basis for R^n in column vector notation.

$B = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} =$

THEOREM 3.26: STANDARD MATRIX FOR A LINEAR TRANSFORMATION

Let $T: R^n \to R^m$ be a linear transformation such that, for the standard basis vectors $\mathbf{e}_i$ of R^n,

$$T(\mathbf{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix},\ T(\mathbf{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix},\ \ldots,\ T(\mathbf{e}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

then the $m \times n$ matrix whose n columns correspond to $T(\mathbf{e}_i)$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in R^n. A is called the standard matrix for T.

Example 5: Find the standard matrix for the linear transformation T.

$$T(x, y) = (4x + y, 0, 2x - 3y)$$

Example 2: Use the standard matrix for the linear transformation T to find the image of the vector $\mathbf{v}$.

$$T(x, y) = (x + y, x - y, 2x, 2y),\ \mathbf{v} = (3, -3)$$

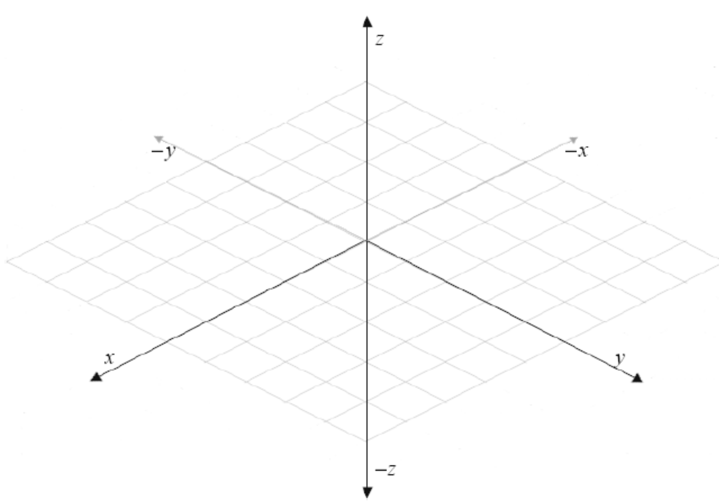
Example 6: Consider the following linear transformation T:

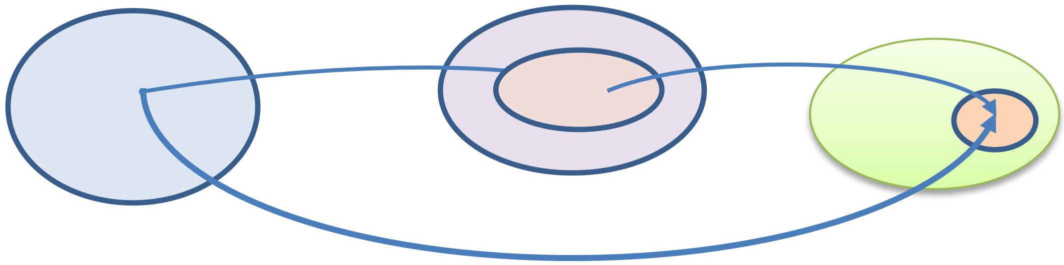
T is the reflection through the *yz*-coordinate plane in $R^3 : T(x, y, z) = (-x, y, z)$, $\mathbf{v} = (2, 3, 4)$.

a. Find the standard matrix A for the following linear transformation T.

b. Use A to find the image of the vector $\mathbf{v}$.

c. Sketch the graph of $\mathbf{v}$ and its image.





THEOREM 3.27: COMPOSITION OF LINEAR TRANSFORMATIONS

Let $T_1 : R^n \to R^m$ and $T_2 : R^m \to R^p$ be linear transformations with standard matrices A_1 and A_2, respectively. The composition $T : R^n \to R^p$, defined by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is a linear transformation. Moreover, the standard matrix A for T is given by the matrix product $A = A_2 A_1$.

Proof:

Example 7: Find the standard matrices A and A' for $T = T_2 \circ T_1$ and $T = T_1 \circ T_2$.

$T_1 : R^2 \to R^3,\ T_1(x, y) = (x, y, y)$

$T_2 : R^3 \to R^2,\ T_2(x, y, z) = (y, z)$

DEFINITION OF INVERSE LINEAR TRANSFORMATION

If $T_1 : R^n \to R^n$ and $T_2 : R^n \to R^n$ are linear transformations such that for every $\mathbf{v}$ in R^n,

then T_2 is called the __________________ of T_1, and T_1 is said to be __________________.

**Not every _____________ transformation has an ________________. If ____ is __________________, however, the inverse is ________________ and is denoted by ______________.

THEOREM 3.28

Let is $T : R^n \to R^n$ be a linear transformation with a standard matrix A. Then the following conditions are equivalent.

1. T is _____________________.
2. T is an ____________________________.
3. A is _______________________.
4. If T is invertible with standard matrix A, then the standard matrix for _______ is _________.

Example 8: Determine whether the linear transformation $T(x, y) = (x + y, x - y)$ is invertible. If it is, find its inverse.

THEOREM 3.29: TRANSFORMATION MATRIX FOR NONSTANDARD BASES

Let V and W be finite-dimensional vector spaces with bases B and B', respectively, where $B=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$.

If $T:V\to W$ is a linear transformation such that

$$[T(\mathbf{v}_1)]_{B'}=\begin{bmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{bmatrix},\ [T(\mathbf{v}_2)]_{B'}=\begin{bmatrix}a_{12}\\a_{22}\\\vdots\\a_{m2}\end{bmatrix},\ \ldots,\ [T(\mathbf{v}_n)]_{B'}=\begin{bmatrix}a_{1n}\\a_{2n}\\\vdots\\a_{mn}\end{bmatrix},$$

then the $m\times n$ matrix whose n columns correspond to $[T(\mathbf{v}_1)]_{B'}$

$$A=\begin{bmatrix}a_{11}&\cdots&a_{1n}\\\vdots&\vdots&\vdots\\a_{m1}&\cdots&a_{mn}\end{bmatrix}$$

Example 9: Find $T(\mathbf{v})$ by using (a) the standard matrix, and (b) the matrix relative to B and B'.

$T:R^3\to R^2,\ T(x,y,z)=(x-y,y-z),\ \mathbf{v}=(1,2,3),$

$B=\{(1,1,1),(1,1,0),(0,1,1)\},\ B'=\{(1,2),(1,1)\}$

Example 10: Let $B=\{e^{2x}, xe^{2x}, x^2e^{2x}\}$ be a basis for a subspace of W of the space of continuous functions, and let D_x be the differential operator on W. Find the matrix for D_x relative to the basis B.

A classical problem in linear algebra is determining whether it is possible to find a basis _____ such that the matrix for _____ relative to _____ is ______________.

1. Matrix for T relative to B: ________
2. Matrix for T relative to B': ________
3. Transition matrix from B' to B: ________
4. Transition matrix from B to B': ________

Example 11: Find the matrix A' relative to the basis B'.

$T: R^2 \to R^2,\ T(x,y) = (x-2y, 4x),\ B' = \{(-2,1),(-1,1)\}$

Example 12: Let $B=\{(1,-1),(-2,1)\}$ and $B'=\{(-1,1),(1,2)\}$ be bases for R^2, $[\mathbf{v}]_{B'}=[1 \quad -4]^T$, and let $A=\begin{bmatrix}2 & 1\\ 0 & -1\end{bmatrix}$ be the matrix for $T:R^2\to R^2$ relative to B.

a. Find the transition matrix P from B' to B.

b. Use the matrices P and A to find $[\mathbf{v}]_B$ and $\left[T(\mathbf{v})_{B'}\right]$ where $[\mathbf{v}]_{B'}=[1 \quad -4]^T$.

DEFINITION OF SIMILAR MATRICES

For square matrices A and A' of order n, A' is said to be similar to A when there exists an invertible matrix P such that $A' = P^{-1}AP$.

THEOREM 3.30

Let A, B, and C be square matrices of order n. Then the following properties are true.

1. A is ______________________ to ______.

2. If A is similar to B, then _____ is ________________ to ____.

3. If A is similar to B and B is similar to C, then _____ is __________________to _____.

Proof:

Example 13: Use the matrix P to show that A and A' are similar.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, A' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}$$

DIAGONAL MATRICES

Diagonal matrices have many ___________________________ advantages over nondiagonal matrices.

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix} \qquad D^k = \begin{pmatrix} \_\_ & 0 & \cdots & 0 \\ 0 & \_\_ & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \_\_ \end{pmatrix}$$

Also, a diagonal matrix is its own _____________________. Additionally, if all the diagonal elements are nonzero, then the inverse of a diagonal matrix is the matrix whose main diagonal elements are the ____________________ of corresponding elements in the original matrix. Because of these advantages, it is important to find ways (if possible) to choose a basis for _____ such that the ___________________________ matrix is ___________________.

Example 14: Suppose $A = \begin{bmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{5}{2} \end{bmatrix}$ is the matrix for $T: R^3 \to R^3$ relative to the standard basis.

Find the diagonal matrix A' for T relative to the basis $B' = \{(1,1,-1),(1,-1,1),(-1,1,1)\}$.

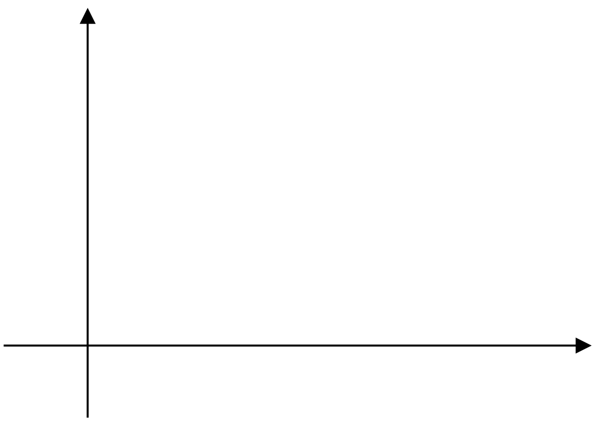
Example 15: Prove that if A is idempotent and B is similar to A, then B is idempotent. (An $n \times n$ matrix is idempotent when $A = A^2$).

Proof:

4.1: INNER PRODUCT SPACES

Learning Objectives:

1. Find the length of a vector and find a unit vector
2. Find the distance between two vectors
3. Find a dot product and the angle between two vectors, determine orthogonality, and verify the Cauchy-Schwartz Inequality, the triangle inequality, and the Pythagorean Theorem
4. Use a matrix product to represent a dot product
5. Determine whether a function defines an inner product, and find the inner product of two vectors in R^n, $M_{m,n}$, P_n, and $C[a,b]$
6. Find an orthogonal projection of a vector onto another vector in an inner product space



DEFINITION OF LENGTH OF A VECTOR IN R^n

The ____________________, _________________, or ____________________________ of a vector $\mathbf{v} = \{v_1, v_2, \ldots, v_n\}$

in ______ is given by

When would the length of a vector equal to 0?

Example 1: Consider the following vectors:

$\mathbf{u} = \left(1, \frac{1}{2}\right)$ $\mathbf{v} = \left(2, -\frac{1}{2}\right)$

a. Find $\|\mathbf{u}\|$

b. Find $\|\mathbf{v}\|$

c. Find $\|\mathbf{u}\|+\|\mathbf{v}\|$

d. Find $\|\mathbf{u}+\mathbf{v}\|$

e. Find $\|3\mathbf{u}\|$

f. Find $3\|\mathbf{u}\|$

Any observations?

THEOREM 4.1: LENGTH OF A SCALAR MULTIPLE

Let $\mathbf{v}$ be a vector in R^n and let c be a scalar. Then

where _____ is the _____________ ______________ of c.

Proof:

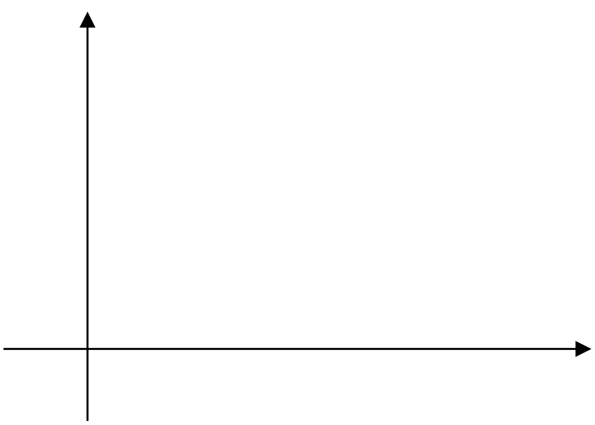
THEOREM 4.2: UNIT VECTOR IN THE DIRECTION OF $\mathbf{v}$

If $\mathbf{v}$ is a nonzero vector in R^n, then the vector

has length _____ and has the same ________________ as $\mathbf{v}$.

Proof:

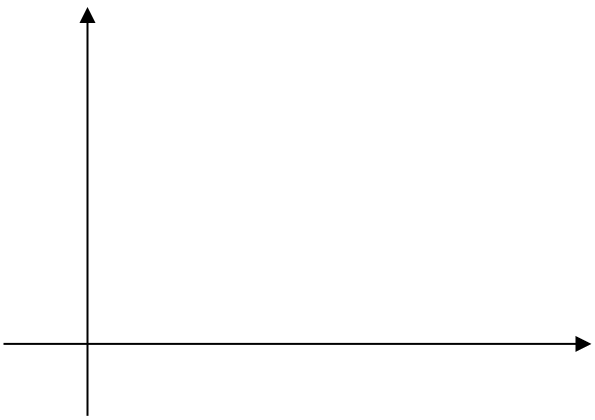
Example 2: Find the vector $\mathbf{v}$ with $\|\mathbf{v}\| = 3$ and the same direction as $\mathbf{u} = (0, 2, 1, -1)$.



DEFINITION OF DISTANCE BETWEEN TWO VECTORS

The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n is

Example 3: Find the distance between $\mathbf{u}=(1,1,2)$ and $\mathbf{v}=(-1,3,0)$.



DEFINITION OF DOT PRODUCT IN R^n

The dot product of $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ and $\mathbf{v}=(v_1,v_2,\ldots,v_n)$ is the ______________ quantity

DEFINITION OF THE ANGLE BETWEEN TWO VECTORS IN R^n

The _____________ _____ between two nonzero vectors in R^n is given by

Example 4: Find the angle between $\mathbf{u} = (2, -1, 1)$ and $\mathbf{v} = (3, 0, 1)$.

Example 5: Consider the following vectors:

$\mathbf{u}=(-1,2)$ $\mathbf{v}=(2,-2)$

a. Find $\mathbf{u}\cdot\mathbf{v}$

b. Find $\mathbf{v}\cdot\mathbf{v}$

c. Find $\|\mathbf{u}\|^2$

d. Find $(\mathbf{u}\cdot\mathbf{v})\mathbf{v}$

e. Find $\mathbf{u}\cdot(5\mathbf{v})$

THEOREM 4.3: PROPERTIES OF THE DOT PRODUCT

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in R^n, and c is a scalar, then the following properties are true.

1. $\mathbf{u}\cdot\mathbf{v} =$ ____
2. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) =$ ____
3. $c(\mathbf{u}\cdot\mathbf{v}) =$ ___________________ $=$ ____________________
4. $\mathbf{v}\cdot\mathbf{v} =$ ___________________
5. $\mathbf{v}\cdot\mathbf{v} \geq 0$ and $\mathbf{v}\cdot\mathbf{v} = 0$ iff _________________________.

Example 6: Find $(3\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-3\mathbf{v})$ given that $\mathbf{u}\cdot\mathbf{u} = 8$, $\mathbf{u}\cdot\mathbf{v} = 7$, and $\mathbf{v}\cdot\mathbf{v} = 6$.

THEOREM 4.4: THE CAUCHY-SCWARZ INEQUALITY

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then

where _____________ denotes the ________________ value of $\mathbf{u}\cdot\mathbf{v}$.

Proof:

Example 7: Verify the Cauch-Schwarz Inequality for $\mathbf{u} = (-1, 0)$ and $\mathbf{v} = (1,1)$.

DEFINITION OF ORTHOGONAL VECTORS

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are orthogonal if

Example 7: Determine all vectors in R^2 that are orthogonal to $\mathbf{u} = (3,1)$.

THEOREM 4.5: THE TRIANGLE INEQUALITY

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then

Proof:

THEOREM 4.6: THE PYTHAGOREAN THEOREM

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

Example 8: Verify the Pythagoren Theorem for the vectors $\mathbf{u} = (3, -2)$ and $\mathbf{v} = (4, 6)$.

DEFINITION OF AN INNER PRODUCT

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in a vector space V, and let c be any scalar. An inner product on V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ and satisfies the following axioms.

1. $\langle \mathbf{u}, \mathbf{v} \rangle =$ ____________________
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle =$ ____________________________
3. $c\langle \mathbf{u}, \mathbf{v} \rangle =$ ____________________
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0, \text{ and } \langle \mathbf{v}, \mathbf{v} \rangle = 0 \text{ iff}$ _______________

NOTE: The __________ product is the _____________________ ________________ product for _______.

Example 8: Show that the function $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2 + u_3v_3$ defines an inner product on R^3, where , $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$.

Example 9: Show that the function $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 - u_2v_2 - u_3v_3$ does not define an inner product on R^3, where , $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$.

THEOREM 4.7: PROPERTIES OF INNER PRODUCTS

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in an inner product space V, and let c be any real number.

1. $\langle \mathbf{0}, \mathbf{v} \rangle =$ __________ $=$ ___________

2. $\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle =$ ______________________

Proof:

3. $\langle \mathbf{u}, c\mathbf{v} \rangle =$ ______________________

DEFINITION OF LENGTH, DISTANCE, AND ANGLE

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V.

1. The length (or ______________) of $\mathbf{u}$ is ________________________.
2. The distance between $\mathbf{u}$ and $\mathbf{v}$ is _______________________________.
3. The angle between and two vectors $\mathbf{u}$ and $\mathbf{v}$ is given by

   ____.
4. $\mathbf{u}$ and $\mathbf{v}$ are orthogonal when __________________________________.

If ________________, then $\mathbf{u}$ is called a ____________ vector. Moreover, if $\mathbf{v}$ is any nonzero vector in an

inner product space V, then the vector ________________________ is a ______________ vector and is

called the _________________ vector in the ___________________ of $\mathbf{v}$.

Inner product on $C[a,b]$ is $\langle f, g \rangle =$ ______________________________________ .

Inner product on $M_{2,2}$ is $\langle A, B \rangle =$ _______________________________________ .

Inner product on P_n is $\langle pq \rangle =$ ____ , where

__________________________________ and __________________________________.

Example 10: Consider the following inner product defined on R^n:

$\mathbf{u} = (0, -6)$, $\mathbf{v} = (-1, 1)$, and $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + 2u_2 v_2$

a. Find $\langle \mathbf{u}, \mathbf{v} \rangle$

b. Find $\|\mathbf{u}\|$

c. Find $\|\mathbf{v}\|$

d. Find $d(\mathbf{u}, \mathbf{v})$

Example 11: Consider the following inner product defined:

$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$, $f(x) = -x$, $g(x) = x^2 - x + 2$

a. Find $\langle f, g \rangle$

b. Find $\| f \|$

c. Find $\|g\|$

d. Find $d(f, g)$

THEOREM 4.8

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V.

Cauchy-Schwarz Inequality: ________________________________

Triangle Inequality: ______________________________________

Pythagorean Theorem: $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

____

Example 12: Verify the triangle inequality for $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$, and

$\langle A, B \rangle = a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + a_{22}b_{22}$.

DEFINITION OF ORTHOGONAL PROJECTION

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V, such that $\mathbf{v} \neq \mathbf{0}$. Then the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$ is

THEOREM 5.9: ORTHOGONAL PROJECTION AND DISTANCE

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V, such that $\mathbf{v} \neq \mathbf{0}$. Then

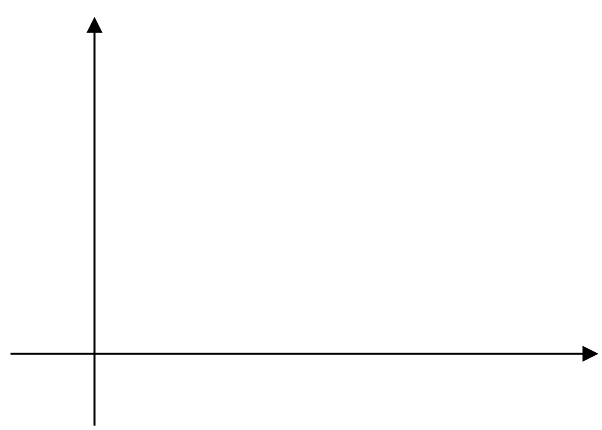
Example 13: Consider the vectors

$\mathbf{u} = (-1, -2)$ and $\mathbf{v} = (4, 2)$. Use the Euclidean inner product to find the following:

a. $\text{proj}_{\mathbf{v}}\mathbf{u}$

b. $\text{proj}_{\mathbf{u}}\mathbf{v}$

c. Sketch the graph of both $\text{proj}_{\mathbf{v}}\mathbf{u}$ and $\text{proj}_{\mathbf{u}}\mathbf{v}$.



4.2: ORTHONORMAL BASES: GRAM-SCHMIDT PROCESS

Learning Objectives:

1. Show that a set of vectors is orthogonal and forms an orthonormal basis, and represent a vector relative to an orthonormal basis
2. Apply the Gram-Schmidt orthonormalization process

Consider the standard basis for R^3, which is

This set is the standard basis because it has useful characteristics such as…The three vectors in the basis are

_______________ __________________, and they are each ___________________ _______________________.

DEFINITIONS OF ORTHOGONAL AND ORTHONORMAL SETS

A set S of a vector space V is called orthogonal when every pair of vectors in S is orthogonal. If, in addition, each vector in the set is a unit vector, then S is called ______________________________. For $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, this definition has the following form.

ORTHOGONAL

ORTHONORMAL

If _____ is a ________________, then it is an ____________________ basis or an ________________________ basis, respectively.

THEOREM 4.10: ORTHOGONAL SETS ARE LINEARLY INDEPENDENT

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of _________________ vectors in an inner product space V, then S is linearly independent.

Proof:

THEOREM 4.10: COROLLARY

If V is an inner product space of dimension n, then any orthogonal set of n nonzero vectors is a basis for V.

Example 1: Consider the following set in R^4.

$$\left\{\left(\frac{\sqrt{10}}{10},0,0,\frac{3\sqrt{10}}{10}\right),(0,0,1,0),(0,1,0,0),\left(-\frac{3\sqrt{10}}{10},0,0,\frac{\sqrt{10}}{10}\right)\right\}$$

a. Determine whether the set of vectors is orthogonal.

b. If the set is orthogonal, then determine whether it is also orthonormal.

c. Determine whether the set is a basis for R^n.

THEOREM 4.11: COORDINATES RELATIVE TO AN ORTHONORMAL BASIS

If $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, then the coordinate representation of a vector $\mathbf{w}$ relative to B is

Proof:

The coordinates of ______ relative to the ____________________ basis ____ are called the

___________________ coefficients of ____ relative to ____. The corresponding coordinate matrix of ___

relative to ____ is

Example 2: Show that the set of vectors $\{(2,-5),(10,4)\}$ in R^2 is orthogonal and normalize the set to produce an orthonormal set.

Example 3: Find the coordinate matrix of $\mathbf{x}=(-3,4)$ relative to the orthonormal basis $B=\left\{\left(\frac{\sqrt{5}}{5},\frac{2\sqrt{5}}{5}\right),\left(-\frac{2\sqrt{5}}{5},\frac{\sqrt{5}}{5}\right)\right\}$ in R^2. Use the dot product as the inner product.

THEOREM 4.12: GRAM-SCHMIDT ORTHONORMALIZATION PROCESS

Let $B=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ be a basis for an inner product V.

Let $B'=\{\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_n\}$, where $\mathbf{w}_i$ is given by

$$\mathbf{w}_1=\mathbf{v}_1$$

$$\mathbf{w}_2=\mathbf{v}_2-\frac{\langle \mathbf{v}_2,\mathbf{w}_1\rangle}{\langle \mathbf{w}_1,\mathbf{w}_1\rangle}\mathbf{w}_1$$

$$\mathbf{w}_3=\mathbf{v}_3-\frac{\langle \mathbf{v}_3,\mathbf{w}_1\rangle}{\langle \mathbf{w}_1,\mathbf{w}_1\rangle}\mathbf{w}_1-\frac{\langle \mathbf{v}_3,\mathbf{w}_2\rangle}{\langle \mathbf{w}_2,\mathbf{w}_2\rangle}\mathbf{w}_2$$

$$\vdots$$

$$\mathbf{w}_n=\mathbf{v}_n-\frac{\langle \mathbf{v}_n,\mathbf{w}_1\rangle}{\langle \mathbf{w}_1,\mathbf{w}_1\rangle}\mathbf{w}_1-\frac{\langle \mathbf{v}_n,\mathbf{w}_2\rangle}{\langle \mathbf{w}_2,\mathbf{w}_2\rangle}\mathbf{w}_2-\cdots-\frac{\langle \mathbf{v}_n,\mathbf{w}_{n-1}\rangle}{\langle \mathbf{w}_{n-1},\mathbf{w}_{n-1}\rangle}\mathbf{w}_{n-1}$$

Let $\mathbf{u}_i=\frac{\mathbf{w}_i}{\|\mathbf{w}_i\|}$. Then the set $B''=\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n\}$ is an orthonormal basis for V. Moreover,

$\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}=\operatorname{span}\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_k\}$ for $k=1,2,\ldots,n$.

Example 4: Apply the Gram-Schmidt orthonormalization process to transform the basis $B=\{(1,0,0),(1,1,1),(1,1,-1)\}$ for a subspace in R^3 into an orthonormal basis. Use the Euclidean inner product on R^3 and use the vectors in the order they are given.

4.3: MATHEMATICAL MODELS AND LEAST SQUARES ANALYSIS

Learning Objectives:

1. When you are done with your homework you should be able to…
2. Define the least squares problem
3. Find the orthogonal complement of a subspace and the projection of a vector onto a subspace
4. Find the four fundamental subspaces of a matrix
5. Solve a least squares problem
6. Use least squares for mathematical modeling

In this section we will study _____________________ systems of linear equations and learn how to find the

_________________ _________________ ____________________ of such a system.

LEAST SQUARES PROBLEM

Given an $m \times n$ matrix A and a vector $\mathbf{b}$ in R^m, the ________________ ________________ problem is to

find ____ in R^m such that __________________ is ______________________.

DEFINITION OF ORTHOGONAL SUBSPACES

The subspaces S_1 and S_2 of R^n are orthogonal when __________________ for all $\mathbf{v}_1$ in S_1 and $\mathbf{v}_2$ in S_2.

Example 1: Are the following subspaces orthogonal?

$$S_1 = \text{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ and } S_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

DEFINITION OF ORTHOGONAL COMPLEMENT

If S is a subspace of R^n, then the orthogonal complement of S is the set

What's the orthogonal complement of $\{\mathbf{0}\}$ in R^n ?

What's the orthogonal complement of R^n ?

DEFINITION OF DIRECT SUM

Let S_1 and S_2 be two subspaces of R^n. If each vector ________________ can be uniquely written as the sum of a vector ____ from ____ and a vector ___ from _____, ____________________, then _____ is the direct sum of ____ and ____, and you can write ____________________.

Example 2: Find the orthogonal complement $S^\perp$, and find the direct sum $S \oplus S^\perp$.

$$S = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right\}$$

THEOREM 4.13: PROPERTIES OF ORTHOGONAL SUBSPACES

Let S be a subspace of R^n , Then the following properties are true.

1. ___________________________________

2. ___________________________________

3. ___________________________________

THEOREM 4.14: PROJECTION ONTO A SUBSPACE

If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_t\}$ is an orthonormal basis for the subspace S of R^n , and $\mathbf{v} \in R^n$, then

Example 3: Find the projection of the vector $\mathbf{v}$ onto the subspace S .

$$S = \text{span}\left\{\begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

THEOREM 4.15: ORTHOGONAL PROJECTION AND DISTANCE

Let S be a subspace of R^n and let $\mathbf{v} \in R^n$. Then, for all $\mathbf{u} \in S$, $\mathbf{u} \neq \text{proj}_S \mathbf{v}$,

FUNDAMENTAL SUBSPACES OF A MATRIX

Recall that if A is an $m \times n$ matrix, then the column space of A is a ________________ of _____ consisting of all vectors of the form _____, _________. The four fundamental subspaces of the matrix A are defined as follows.

__________ = nullspace of A

__________ = nullspace of A^T

__________ = column space of A

__________ = column space of A^T

Example 4: Find bases for the four fundamental subspaces of the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

THEOREM 4.16: FUNDAMENTAL SUBSPACES OF A MATRIX

If A is an $m \times n$ matrix, then

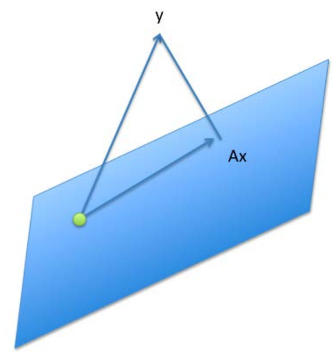
__________ and ___________ are orthogonal subspaces of _____.

__________ and ___________ are orthogonal subspaces of _____.

____

____

SOLVING THE LEAST SQUARES PROBLEM



Recall that we are attempting to find a vector $\mathbf{x}$ that minimizes ______________,

where A is an $m \times n$ matrix and $\mathbf{b}$ is a vector in R^m. Let S be the column space of A: ________________. Assume that $\mathbf{b}$ is not in S, because otherwise the system $A\mathbf{x} = \mathbf{b}$ would be _________________________. We are looking for a vector ______ in ____ that is as close as possible to ___. This desired vector is the ____________________ of _____ onto ____. So, ___________________ and ________________________ = _______________________ is orthogonal to _________________. However, this implies that ______________ is in __________, which equals ________________. So, _____________ is in the __________________ of _____.

The solution of the least squares problem comes down to solving the __________ linear system of equations _______________________. These equations are called the _________________ equations of the least squares problem _______________.

Example 5: Find the least squares solution of the system $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

Example 6: The table shows the numbers of doctoral degrees y (in thousands) awarded in the United States from 2005 through 2008. Find the least squares regression line for the data. Then use the model to predict the number of degrees awarded in 2015. Let t represent the year, with $t = 5$ corresponding to 2005. (Source: U.S. National Center for Education Statistics)

Year	**2005**	**2006**	**2007**	**2008**
Doctoral Degrees, *y*	52.6	56.1	60.6	63.7

4.4: EIGENVALUES AND EIGENVECTORS, AND DIAGONALIZING MATRICES

Learning Objectives:

1. Verify eigenvalues and corresponding eigenvectors
2. Find eigenvectors and corresponding eigenspaces
3. Use the characteristic equation to find eigenvalues and eigenvectors, and find the eigenvalues and eigenvectors of a triangular matrix
4. Find the eigenvalues and eigenvectors of a linear transformation

THE EIGENVALUE PROBLEM

One of the most important problems in linear algebra is the **eigenvalue problem**. When A is an $n \times n$, do nonzero vectors $\mathbf{x}$ in R^n exist such that $A\mathbf{x}$ is a ____________ multiple of $\mathbf{x}$? The scalar, denoted by _____ (___________), is called an ___________________ of the matrix A, and the nonzero vector $\mathbf{x}$ is called an ___________________ of A corresponding to λ.

DEFINITIONS OF EIGENVALUE AND EIGENVECTOR

Let A be an $n \times n$ matrix. The scalar ____ is called an _________________ of A when there is a ____________ vector $\mathbf{x}$ such that ____________. The vector $\mathbf{x}$ is called an _________________ of A corresponding to λ.

*Note that an eigenvector cannot be _____. Why not?

Example 1: Determine whether $\mathbf{x}$ is an eigenvector of A.

$$A = \begin{bmatrix} -3 & 10 \\ 5 & 2 \end{bmatrix}$$

a. $\mathbf{x} = (-8, 4)$

b. $\mathbf{x} = (5, -3)$

THEOREM 4.17: EIGENVECTORS OF λ FORM A SUBSPACE

If A is an $n \times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ, together with the zero vector

is a subspace of R^n. This subspace is called the ________________ of λ.

Proof:

THEOREM 4.18: EIGENVALUES AND EIGENVECTORS OF A MATRIX

Let A be an $n \times n$ matrix.

1. An eigenvalue of A is a scalar λ such that _______________.
2. The eigenvectors of A corresponding to λ are the ____________solutions of

   ____________________.

* The equation _________________ is called the _____________________ _____________________ of A. When expanded to polynomial form, the polynomial is called the ________________ ___________________ of A. This definition tells you that the ___________________ of an $n \times n$ matrix A correspond to the ____________ of the characteristic polynomial of A.

Example 2: Find (a) the characteristic equation and (b) the eigenvalues (and corresponding eigenvectors) of the matrix.

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

THEOREM 4.19: EIGENVALUES OF TRIANGULAR MATRICES

If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main ________________________.

Example 3: Find the eigenvalues of the triangular matrix.

$$\begin{bmatrix} -5 & 0 & 0 \\ 3 & 7 & 0 \\ 4 & -2 & 3 \end{bmatrix}$$

EIGENVALUES AND EIGENVECTORS OF LINEAR TRANSFORMATIONS

A number λ is called an ______________________ of a linear transformation _____________ when there is a

________________ vector ____ such that _________________. The vector $\mathbf{x}$ is called an ___________________

of T corresponding to λ, and the set of all eigenvectors of λ (with the zero vector) is called the

__________________ of λ.

Example 4: Consider the linear transformation $T: R^n \to R^n$ whose matrix A relative to the standard base is given. Find (a) the eigenvalues of A, (b) a basis for each of the corresponding eigenspaces, and (c) the matrix A' for T relative to the basis B', where B' is made up of the basis vectors found in part b).

$$A = \begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix}$$

4.5: DIAGONALIZATION

Learning Objectives:

1. Find the eigenvectors of similar matrices, determine whether a matrix A is diagonalizable, and find a matrix P such that $P^{-1}AP$ is diagonal
2. Find, for a linear transformation $T : V \to V$, a basis B for V such that the matrix for T relative to B is diagonal

DEFINITION OF A DIAGONALIZABLE MATRIX

An $n \times n$ matrix A is diagonalizable when A is similar to a diagonal matrix. That is, A is diagonalizable when there exists an invertible matrix ____ such that _____________ is a diagonal matrix.

THEOREM 4.20: SIMILAR MATRICES HAVE THE SAME EIGENVALUES

If A and B are similar $n \times n$ matrices, then the have the same ____________________________.

Proof:

Example 1: (a) verify that A is diagonalizable by computing $P^{-1}AP$, and (b) use the result of part (a) and Theorem 4.20 to find the eigenvalues of A.

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}, \ P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

THEOREM 4.21: CONDITION FOR DIAGONALIZATION

An $n \times n$ matrix A is diagonalizable if and only if it has n __________________ ______________________ eigenvectors.

Proof:

Example 2: For the matrix A, find, if possible, a nonsingular matrix P such that $P^{-1}AP$ is diagonal. Verify $P^{-1}AP$ is a diagonal matrix with the eigenvalues on the main diagonal.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

STEPS FOR DIAGONALIZING AN $n \times n$ SQUARE MATRIX

Let A be an $n \times n$ matrix.

1. Find n linearly independent eigenvectors _______________________ for A (if possible) with corresponding eigenvalues ______________________. If n linearly independent eigenvectors do not exist, then A is not diagonalizable.
2. Let P be the $n \times n$ matrix whose columns consist of these eigenvectors. That is, _________________________. The diagonal matrix __________________ will have the eigenvalues _____________________ on its main ___________________ (and __________ elsewhere). Note that the order of the eigenvectors used to form P will determine the order in which the eigenvalues appear on the main __________________ of _____.

THEOREM 4.22: SUFFICIENT CONDITION FOR DIAGONALIZATION

If an $n \times n$ matrix A has ____ _____________ eigenvalues, then the corresponding eigenvectors are __________________ __________________ and A is ________________________.

Proof:

Example 3: Find the eigenvalues of the matrix and determine whether there is a sufficient number to guarantee that the matrix is diagonalizable.

$$\begin{bmatrix} 2 & 0 \\ 5 & 2 \end{bmatrix}$$

Example 4: Find a basis B for the domain of T such that the matrix for T relative to B is diagonal.

$$T: R^3 \rightarrow R^3 : T(x, y, z) = (-2x + 2y - 3z, 2x + y - 6z, -x - 2y)$$

4.5: SYMMETRIC MATRICES AND ORTHOGONAL DIAGONALIZATION

Learning Objectives:

1. Recognize, and apply properties of, symmetric matrices
2. Recognize, and apply properties of, orthogonal matrices
3. Find an orthogonal matrix P that orthogonally diagonalizes a symmetric matrix A

SYMMETRIC MATRICES

Symmetric matrices arise more often in ________________________ than any other major class of matrices.

The theory depends on both ___________________________ and _____________________________. For

most matrices, you need to go through most of the diagonalization ___________ to ascertain whether a

matrix is _______________________. We learned about one exception, a ____________________ matrix,

which has ____________ entries on the main ________________________. Another type of matrix which

is guaranteed to be ________________________ is a ____________________ matrix.

DEFINITION OF SYMMETRIC MATRIX

A square matrix A is ___________________ when it is equal to its _________________:________________.

Example 1: Determine which of the matrices below are symmetric.

$$A=\begin{bmatrix}-2 & 5\\ 5 & 1\end{bmatrix},\ B=\begin{bmatrix}6 & 5 & 4\\ 5 & 1 & 0\\ 4 & 0 & -1\end{bmatrix},\ C=\begin{bmatrix}3 & 2 & 1\\ 1 & 2 & 3\end{bmatrix},\ D=\begin{bmatrix}1 & 2 & 3 & 4\\ 2 & 7 & 1 & 0\\ 3 & 1 & 7 & 2\\ 4 & 0 & 2 & 5\end{bmatrix}$$

Example 2: Using the diagonalization process, determine if A is diagonalizable. If so, diagonalize the matrix A.

$$A = \begin{bmatrix} 6 & -1 \\ -1 & 5 \end{bmatrix}$$

THEOREM 4.23: PROPERTIES OF SYMMETRIC MATRICES

If A is an $n \times n$ symmetric matrix, then the following properties are true.

1. A is ______________________________.

2. All _________________________ of A are __________.

3. If λ is an _____________________ of A with multiplicity ____, then

   ____ has ____ linearly __________________ eigenvectors. That is, the

   _________________ of λ has dimension ____.

Proof of Property 1 (for a 2 x 2 symmetric matrix):

Example 3: Prove that the symmetric matrix is diagonalizable.

$$A = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$$

Example 4: Find the eigenvalues of the symmetric matrix. For each eigenvalue, find the dimension of the corresponding eigenspace.

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

DEFINITION OF AN ORTHOGONAL MATRIX

A square matrix P is ___________________ when it is _________________ and when

__________________.

THEOREM 4.24: PROPERTY OF ORTHOGONAL MATRICES

An $n \times n$ matrix P is orthogonal if and only if its ________________________ vectors form an

___________________________ set.

Example 5: Determine whether the matrix is orthogonal. If the matrix is orthogonal, then show that the column vectors of the matrix form an orthonormal set.

$$A = \begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$$

THEOREM 4.25: PROPERTY OF SYMMETRIC MATRICES

Let A be an $n \times n$ symmetric matrix. If λ_1 and λ_2 are _________________ eigenvalues of A, then their corresponding _____________________ $\mathbf{x}_1$ and $\mathbf{x}_2$ are ______________________.

THEOREM 4.26: FUNDAMENTAL THEOREM OF SYMMETRIC MATRICES

Let A be an $n \times n$ matrix. Then A is ______________________________ ________________________ and has _____________ eigenvalues if and only if A is __________________________.

Proof:

STEPS FOR DIAGONALIZING A SYMMETRIC MATRIX

Let A be an $n \times n$ symmetric matrix.

1. Find all _______________________ of A and determine the _______________________ of each.
2. For _____________ eigenvalue of multiplicity _____, find a _________eigenvector. That is, find any ______________________ and then ___________________ it.
3. For _____________ eigenvalue of multiplicity ________, find a set of _____ _____________________ ___________________________ eigenvectors. If this set is not ________________________, apply the _________________________ _____________________________ process.
4. The results of steps 2 and 3 produce an ____________________ set of _______ eigenvectors. Use these eigenvectors to form the __________ of ______. The matrix ____________________________ will be ____________________. The main entries of _____ are the _____________________ of _____.

Example 5: Find a matrix P such that $P^T AP$ orthogonally diagonalizes A. Verify that $P^T AP$ gives the proper diagonal form.

$$A=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Example 6: Prove that if a symmetric matrix A has only one eigenvalue λ, then $A=\lambda I$.

4.6: APPLICATIONS OF EIGENVALUES AND EIGENVECTORS

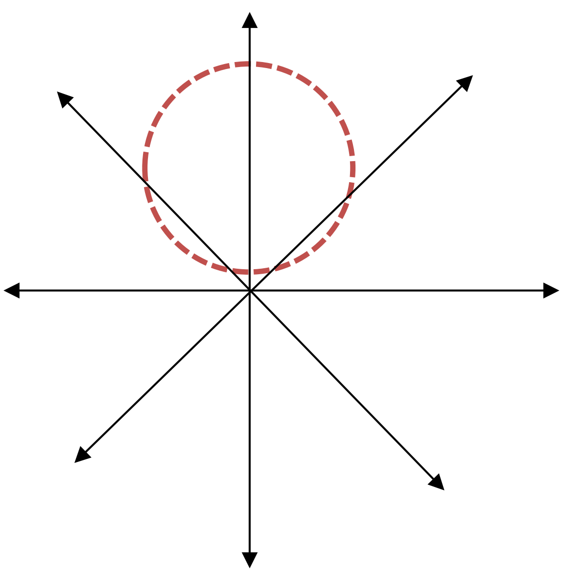
Learning Objectives:

1. Find the matrix of a quadratic form and use the Principal Axes Theorem to perform a rotation of axes for a conic and a quadric

QUADRATIC FORMS

Every conic section in the *xy*-plane can be written as:

If the equation of the conic has no *xy*-term (______________), then the axes of the graphs are parallel to the coordinate axes. For second-degree equations that have an *xy*-term, it is helpful to first perform a ___________________ of axes that eliminates the *xy*-term. The required rotation angle is $\cot 2\theta = \frac{a-c}{b}$. With this rotation, the standard basis for R^2, ______________________ is rotated to form the new basis ______________________________________.

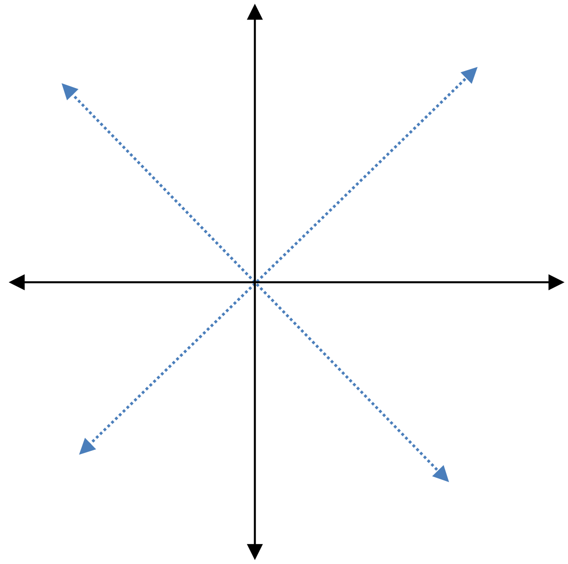


Example 1: Find the coordinates of a point (x, y) in R^2 relative to the basis $B' = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$.

ROTATION OF AXES

The general second-degree equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ can be written in the form $a'(x')^2 + c'(y')^2 + d'x' + e'y' + f' = 0$ by rotating the coordinate axes counterclockwise through the angle θ, where θ is defined by $\cot 2\theta = \frac{a-c}{b}$. The coefficients of the new equation are obtained from the substitutions $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$.

Example 2: Perform a rotation of axes to eliminate the *xy*-terms in
$5x^2 - 6xy + 5y^2 + 14\sqrt{2}x - 2\sqrt{2}y + 18 = 0$. Sketch the graph of the resulting equation.



________________________ and _________________________ can be used to solve the rotation of axes problem. It turns out that the coefficients *a'* and *c'* are eigenvalues of the matrix

The expression _______________________ is called the _________________ form associated with the quadratic equation and the matrix ____ is called the _______________ of the _______________ form. Note that ____ is ____________________. Moreover, ___ will be ________________________ if and only if its corresponding quadratic form has no ______ term.

Example 3: Find the matrix of quadratic form associated with each quadratic equation.

a. $x^2 + 4y^2 + 4 = 0$

b. $5x^2 - 6xy + 5y^2 + 14\sqrt{2}x - 2\sqrt{2}y + 18 = 0$

Now, let's check out how to use the matrix of quadratic form to perform a rotation of axes.

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$. Then the quadratic expression $ax^2 + bxy + cy^2 + dx + ey + f$ can be written in matrix form as follows:

If ______________, then no ____________________ is necessary. But if _____________, then because ___ is symmetric, you may conclude that there exists an ___________________ matrix ____ such that ___________ is diagonal. So, if you let then it follows that __________, and

The choice of ___ must be made with care. Since ___ is orthogonal, its determinant will be __________. If P is chosen so that $|P| = 1$, then P will be of the form

where θ gives the angle of rotation of the conic measured from the __________ *x*-axis to the positive *x′*-axis.

PRINCIPAL AXES THEOREM

For a conic whose equation is $ax^2 + bxy + cy^2 + dx + ey + f = 0$, the rotation given by __________________ eliminates the *xy*-term when P is an orthogonal matrix, with $|P| = 1$, that diagonalizes A. That is

where λ_1 and λ_2 are eigenvalues of A. The equation of the rotated conic is given by

Example 4: Use the Principal Axes Theorem to perform a rotation of axes to eliminate the *xy*-term in the quadratic equation. Identify the resulting rotated conic and give its equation in the new coordinate system.

$5x^2 - 6xy + 5y^2 + 14\sqrt{2}x - 2\sqrt{2}y + 18 = 0$