1/26/11	1/28/11	Monday	Next Wed.
◦ Lecture 5.7	• Start 5.8	Finish 5.8	Review

When you are done with your homework you should be able to…

- π Integrate functions whose antiderivatives involve inverse trigonometric functions
- π Use the method of completing the square to integrate a function
- π Review the basic integration rules involving elementary functions

$$\frac{\partial}{\partial x}(xy) = 1y + x\frac{\partial y}{\partial x}$$
$$\frac{\partial}{\partial x}(x+y) = 1 + \frac{\partial y}{\partial x}$$

Warm-up:

1. Differentiate the following functions with respect to x.

a. $y = \arctan\frac{x}{2} - \frac{1}{2(x^2+4)}$.

$$\frac{dy}{dx} = \frac{\frac{1}{2}}{1+\left(\frac{x}{2}\right)^2} + \frac{1}{\cancel{2}}\left(x^2+4\right)^{-2}\left(\cancel{2}x\right)$$

$$\frac{dy}{dx} = \frac{1}{\cancel{2}} \cdot \frac{1}{\left(\frac{x^2+4}{\cancel{4}\,2}\right)} + \frac{x}{\left(x^2+4\right)^2}$$

$$\frac{dy}{dx} = \frac{2(x^2+4)+x}{(x^2+4)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{2x^2+x+8}{(x^2+4)^2}}$$

b. $\frac{\partial}{\partial x}\arctan(xy) = \frac{\partial}{\partial x}\arcsin(x+y)$.

$$\frac{y + x\frac{\partial y}{\partial x}}{1+x^2y^2} = \frac{1+\frac{\partial y}{\partial x}}{\sqrt{1-(x+y)^2}}$$

Simplification left to student...

2. Complete the square.

a. $3+4x-x^2$

$$-x^2+4x \quad +3$$
$$= -\left(x^2-4x+(-2)^2\right)+3+4$$
$$= \boxed{7-(x-2)^2}$$

b. $2x^2-6x+9$

$$= 2\left(x^2-3x+\left(\tfrac{-3}{2}\right)^2\right)+9-\frac{9}{2}$$
$$= \boxed{\frac{9}{2}+2\left(x-\frac{3}{2}\right)^2}$$

What did you notice about the derivatives of the inverse trigonometric functions?

THEOREM: INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x, and let $a>0$.

1. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a}+C$
2. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a}+C$
3. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a}+C$

Example 1: Find the integral.

a. $\int \frac{dx}{x\sqrt{x^2-1}}$

$u=x$, $du=dx$, $a=1$

$= \frac{1}{1}\operatorname{arcsec}\frac{|x|}{1}+C$

$= \boxed{\operatorname{arcsec}|x|+C}$

b. $\int \frac{xdx}{\sqrt{x^2-1}}$

$u=x^2-1$, $\frac{du}{dx}=2x$, $dx=\frac{du}{2x}$

$= \int \frac{\cancel{x}\left(\frac{du}{2\cancel{x}}\right)}{u^{1/2}}$

$= \cancel{\frac{1}{2}}\left(\frac{u^{1/2}}{\cancel{\frac{1}{2}}}\right)+C$

$\boxed{\sqrt{x^2-1}+C}$

c. $\int \frac{dx}{\sqrt{1-x^2}}$

$= \arcsin\left(\frac{x}{1}\right)+C$

$= \boxed{\arcsin x+C}$

d. $\int \frac{dx}{x\ln x}$

$u=\ln x$

$\frac{du}{dx}=\frac{1}{x} \rightarrow dx=xdu$

$= \int \frac{\cancel{x}du}{\cancel{x}\,u}$

$= \ln|u|+C$

$= \boxed{\ln|\ln x|+C}$

e. $\int \frac{(\ln x)^2\,dx}{x}$

$u=\ln x$

$\frac{du}{dx}=\frac{1}{x}$

$dx=xdu$

$= \int \frac{u^2(\cancel{x}du)}{\cancel{x}}$

$= \frac{u^3}{3}+C$

$= \boxed{\frac{(\ln x)^3}{3}+C}$

f. $\int \ln x\,dx$

Needs integration by parts → 8.2

Example 2: Find the integral by completing the square.

a. $\int \frac{dx}{x^2+4x+13} = \int \frac{dx}{9+(x+2)^2} = \int \frac{du}{3^2+u^2} = \frac{1}{3}\arctan\frac{u}{3}+C$

$= \boxed{\frac{1}{3}\arctan\left(\frac{x+2}{3}\right)+C}$

need to complete the square!

$x^2+4x+(2)^2+13-4$

$(x+2)^2+9$

$a^2=9$

$a=3$

$u=x+2$

$du=dx$

b. $\int \frac{dx}{x^2+4x+13}$

c. $\int \frac{2dx}{\sqrt{-x^2+4x}} = 2\int \frac{dx}{\sqrt{(2)^2-(x-2)^2}} = 2\int \frac{du}{\sqrt{(2)^2-u^2}}$

$= 2\arcsin\left(\frac{u}{2}\right)+C$

$= \boxed{2\arcsin\left(\frac{x-2}{2}\right)+C}$

need to complete square

$-(x^2-4x+(-2)^2)+4$

$4-(x-2)^2$

$a=2$

$u=x-2$

$du=dx$

d. $\int \frac{2x-5+7-7}{x^2+2x+2}dx = \int \frac{2x+2}{x^2+2x+2}dx - 7\int \frac{dx}{x^2+2x+2}$

*complete square

$(x^2+2x+(1)^2)+2-1$

$(x+1)^2+1$

$a=1$

$u=x+1$

$du=dx$

*$u=x^2+2x+2$

$\frac{du}{dx}=2x+2$

$dx=\frac{du}{2x+2}$

$=\int \frac{\cancel{(2x+2)}}{u}\cdot\frac{du}{\cancel{(2x+2)}} - 7\int\frac{du}{1^2+(x+1)^2}$ *

$= \ln|u| - 7\left(\frac{1}{1}\arctan\left(\frac{u}{1}\right)\right)+C$

$= \boxed{\ln|x^2+2x+2| - 7\arctan(x+1)+C}$

e. $\int \frac{x}{\sqrt{9+8x^2-x^4}}dx = \int \frac{\cancel{x}}{\sqrt{(5)^2-u^2}}\cdot\frac{du}{2\cancel{x}} = \boxed{\frac{1}{2}\arcsin\frac{x^2-4}{5}+C}$

complete square

$-(x^4-8x^2+(-4)^2)+9+16$

$25-(x^2-4)^2$

$a^2=25 \rightarrow a=5$

$u=x^2-4$

$\frac{du}{dx}=2x$

$dx=\frac{du}{2x}$

Example 3: Find the area of the region bound by the graphs of

$$y=\frac{4e^x}{1+e^{2x}},\ x=0,\ y=0 \text{ and } x=\ln\sqrt{3}.$$

hint: $e^{2x}=(e^x)^2$

$$A=\int_0^{\ln\sqrt{3}}\frac{4e^x}{1+(e^x)^2}\,dx = 4\int_1^{\sqrt{3}}\frac{\cancel{e^x}}{1+u^2}\cdot\frac{du}{\cancel{e^x}} = \frac{4}{1}\arctan\frac{u}{1}\Big|_1^{\sqrt{3}}$$

$u=e^x$

$\frac{du}{dx}=e^x$

$dx=\frac{du}{e^x}$

limits:

upper: $e^{\ln\sqrt{3}}=\sqrt{3}$

lower: $e^0=1$

$$=4(\arctan\sqrt{3}-\arctan 1)$$

$$=4\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

$$=\cancel{4}^{1}\left(\frac{4\pi-3\pi}{\cancel{12}_{3}}\right)$$

$$=\boxed{\frac{\pi}{3}\text{ sq. units}}$$