For Use With Calculus, 10th Ed. By Ron Larson and Bruce Edwards

# CALCULUS III GUIDED NOTEBOOK

Created by Shannon Martin Myers

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# CHAPTER 11 Vectors and the Geometry of Space

## Section 11.1 Vectors in the Plane

When you are done with your homework you should be able to…

- π Write the component form of a vector
- π Perform vector operations and interpret the results geometrically
- π Write a vector as a linear combination of standard unit vectors
- π Use vectors to solve problems involving force or velocity

Warm-up: Find the distance between the points (2, 1) and (4, 7).

What is a scalar quantity?

Give examples of quantities which can be characterized by a scalar.

What is a vector?

Give examples of quantities which are represented by vectors.

How do you find the length, aka magnitude, aka norm, of a vector?

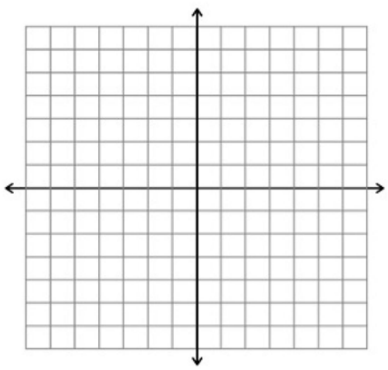
What makes two vectors equivalent?

**DEFINITION OF COMPONENT FORM OF A VECTOR IN THE PLANE**

If $\mathbf{v}$ is a vector in the plane whose initial point is the origin and whose terminal point is $(v_1, v_2)$, then the _________________ __________ of $\mathbf{v}$ is given by

The coordinates $v_1$ and $v_2$ are called the __________________ of $\mathbf{v}$. If both the initial point and the terminal point lie at the origin, then $\mathbf{v}$ is called the _______________ vector and is denoted by $\mathbf{0} = \langle 0, 0 \rangle$.

Example 1: Sketch the vector whose initial point is the origin and whose terminal point is (3, -2).



**DEFINITIONS OF VECTOR ADDITION AND SCALAR MULTIPLICATION**

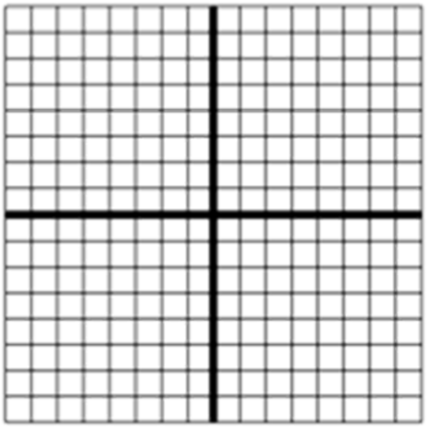
Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and let *c* be a scalar.

1. The _______________ _________ of $\mathbf{u}$ and $\mathbf{v}$ is the vector _______________ = ____________________.
2. The ______________ __________________ of *c* and $\mathbf{u}$ is the vector ________ = ____________________.
3. The ________________ of $\mathbf{v}$ is the vector _________ = ________________ = ______________________.
4. The ___________________ of $\mathbf{u}$ and $\mathbf{v}$ is the vector _______________ = _________________ = ___________________________.

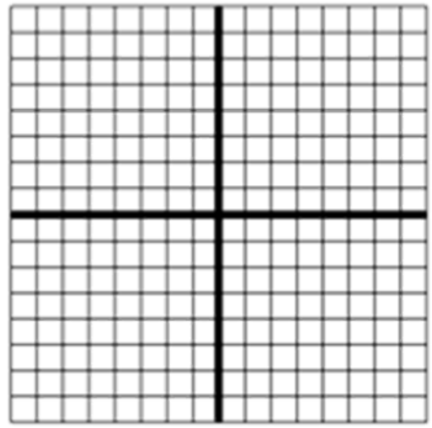
Example 2: Find the component form and length of the vector $\mathbf{v}$ that has initial point (-1, 4) and terminal point (7, 3). Find the norm of $\mathbf{v}$.

Example 3: Let $\mathbf{u} = \langle -1, -3 \rangle$ and $\mathbf{v} = \langle 2, -8 \rangle$ find the following vectors. Illustrate the vector operations geometrically.

a. $\mathbf{u} - \mathbf{v}$



b. $-2\mathbf{v}$



**THEOREM: PROPERTIES OF VECTOR OPERATIONS**

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane, and let *c* and *d* be scalars.

1. $\mathbf{u} + \mathbf{v} =$ ___________
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} =$ ______________
3. $\mathbf{u} + \mathbf{0} =$ _____
4. $\mathbf{u} + (-\mathbf{u}) =$ _____
5. $c(d\mathbf{u}) =$ ______
6. $(c + d)\mathbf{u} =$ ___________
7. $c(\mathbf{u} + \mathbf{v}) =$ ___________
8. $1(\mathbf{u}) =$ ___, and $0(\mathbf{u}) =$ ___

**THEOREM: LENGTH OF A SCALAR MULTIPLE**

Let $\mathbf{v}$ be a vector, and let $c$ be a scalar. Then

**THEOREM: UNIT VECTOR IN THE DIRECTION OF $\mathbf{v}$**

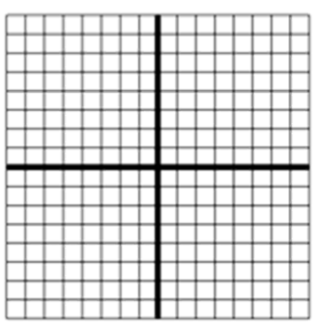
If $\mathbf{v}$ is a nonzero vector in the plane, then the vector

Has length _____ and the same ________________ as **v.**

Example 4: Find a unit vector in the direction of $\mathbf{v} = \langle 7, -5 \rangle$. Verify that it has length 1.

**Standard Unit Vectors**

$\mathbf{i} = \langle \ \ , \ \ \rangle$ and $\mathbf{j} = \langle \ \ , \ \ \rangle$

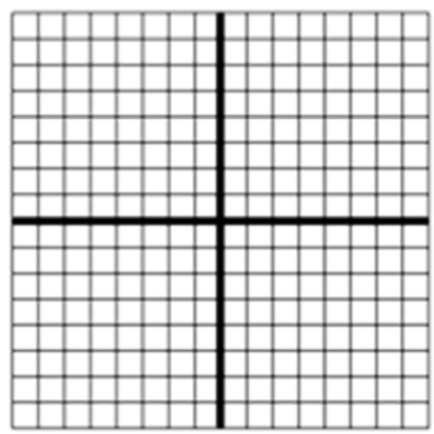


Example 5: Let $\mathbf{u}$ be the vector with initial point (-4, 1) and terminal point (3, -1) and let $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j}$. Write each vector as a linear combination of **i** and **j**.

a. $\mathbf{u}$

b. $\mathbf{w} = 4\mathbf{u} - 2\mathbf{v}$

Example 6: The vector **v** has a magnitude of 2 and makes an angle of $\dfrac{\pi}{3}$ with the positive *x*-axis. Write **v** as a linear combination of the unit vectors **i** and **j**.



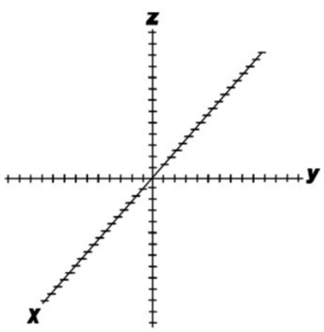
# Section 11.2 Space Coordinates and Vectors In Space

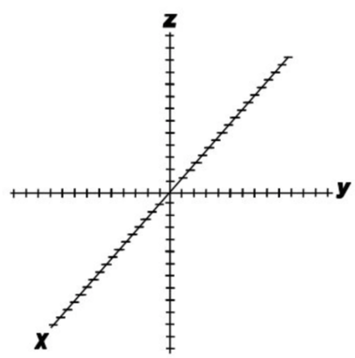
When you are done with your homework you should be able to…

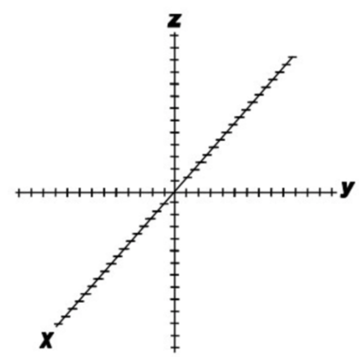
- π Understand the three-dimensional rectangular coordinate system
- π Analyze vectors in space
- π Use three-dimensional vectors to solve real-life problems

Warm-up: Find the vector $\mathbf{v}$ with magnitude 4 and the same direction as $\mathbf{u} = \langle -1, 1 \rangle$.

Constructing a **three-dimensional coordinate system**:

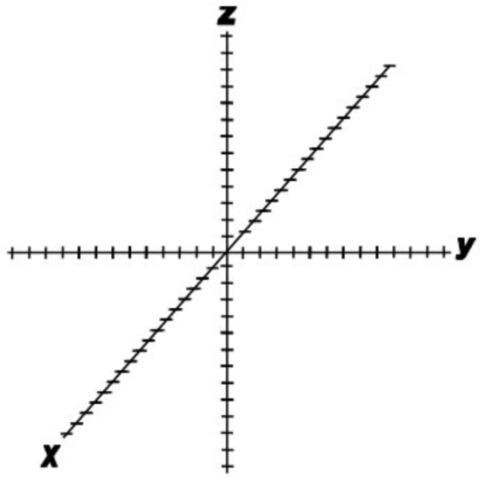






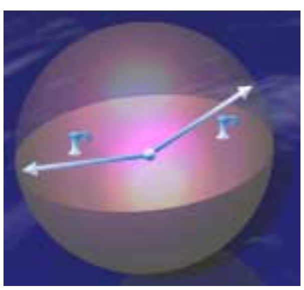

- π Taken as pairs, the axes determine three **coordinate planes**: the *xy*-plane, the *xz*-plane, and the *yz*-plane
    - o These planes separate the three-space into ________ octants
- π In this three-dimensional system, a point *P* in space is determined by an ordered ________________, denoted _____________
    - o *x* = directed distance from *yz*-plane to *P*
    - o *y* = directed distance from *xz*-plane to *P*
    - o *z* = directed distance from *xy*-plane to *P*
- π A three-dimensional coordinate system can either have a **left-handed** or **right-handed** orientation
    - o The right-handed system has the right hand pointing along the *x*-axis
    - o Our text uses the right-handed system

Example 1: Draw a three-dimensional coordinate system and plot the following points: A(1, 0, 4), B(-2, 3, 1) and C(-2, -1, -4)




**THE DISTANCE BETWEEN TWO POINTS IN SPACE**

Example 2: Find the standard equation of the sphere that has the points (0, 1, 3) and (-2, 4, 2) as endpoints of a diameter.



**DEFINITIONS OF VECTOR ADDITION AND SCALAR MULTIPLICATION**

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in space and let *c* be a scalar.

1. *Equality of Vectors*. $\mathbf{u} = \mathbf{v}$ if and only if _________, _________, and _________.
2. *Component Form.* If $\mathbf{v}$ is represented by the directed line segment from $P(p_1, p_2, p_3)$ to $Q(q_1, q_2, q_3)$, then ______________ = ________________________________.
3. *Length*. $\|\mathbf{v}\| =$ ____________________ .
4. *Unit Vector in the Direction of* $\mathbf{v}$. $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} =$ ___________________, $\mathbf{v} \neq \mathbf{0}$.
5. *Vector Addition.* $\mathbf{v} + \mathbf{u} =$ ________________________ .
6. *Scalar Multiplication.* $c\mathbf{v} =$ __________________ .

Example 3: Find the component form of the vector $\mathbf{v}$ that has initial point (-1, 6, 4) and terminal point (0, -5, 3). Find a unit vector in the direction of $\mathbf{v}$.

**DEFINITION: PARALLEL VECTORS**

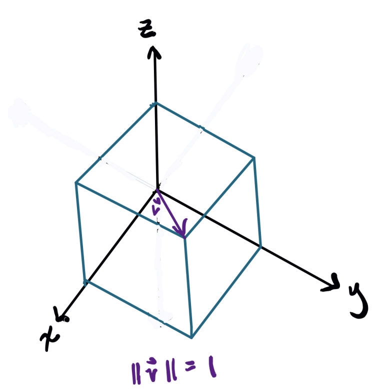
Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if there is some scalar *c* such that

Example 4: Vector $\mathbf{z}$ has initial point (5, 4, 1) and terminal point (-2, -4, 4). Determine which of the vectors is parallel to $\mathbf{z}$.

a. $\langle 7,6,2 \rangle$

b. $\langle 14,16,-6 \rangle$

Example 5: Find the component form of the unit vector $\mathbf{v}$ in the direction of the diagonal of the cube shown in the figure.



# Section 11.3 The Dot Product of Two Vectors

When you are done with your homework you should be able to…

- π Use properties of the dot product of two vectors
- π Find the angle between two vectors using the dot product
- π Find the direction cosines of a vector in space
- π Find the projection of a vector onto another vector
- π Use vectors to find the work done by a constant force

Warm-up: Write the equation of the sphere in standard form. Find the center and the radius.

$$9x^2+9y^2+9z^2-6x+18y+1=0$$

**DEFINITION OF DOT PRODUCT (aka Euclidean inner product aka scalar product)**

The dot product of $\mathbf{u}=\langle u_1,u_2\rangle$ and $\mathbf{v}=\langle v_1,v_2\rangle$ is

The dot product of $\mathbf{u}=\langle u_1,u_2,u_3\rangle$ and $\mathbf{v}=\langle v_1,v_2,v_3\rangle$ is

**THEOREM: PROPERTIES OF THE DOT PRODUCT**

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in the plane or in space and let *c* be a scalar.

1. *Commutative Property*. $\mathbf{u} \cdot \mathbf{v} =$ __________
2. *Distributive Property*. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) =$ _______________
3. $c(\mathbf{u} \cdot \mathbf{v}) =$ ___________ $=$ ____________

   Proof:
4. $\mathbf{0} \cdot \mathbf{v} =$ ____
5. $\mathbf{v} \cdot \mathbf{v} =$ ________

   Proof:

Example 1: Given $\mathbf{u} = \langle -4, 6 \rangle$, $\mathbf{v} = \langle 3, 7 \rangle$ and $\mathbf{w} = \langle 9, -5 \rangle$, find each of the following:

a. $\mathbf{u} \cdot \mathbf{w}$

b. $5\mathbf{u} \cdot \mathbf{v}$

c. $\mathbf{u} \cdot \mathbf{u}$

d. $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$

**THEOREM: ANGLE BETWEEN TWO VECTORS**

If $\theta,\ 0 \le \theta \le \pi,$ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ then

____________ = ________________________ is the _______________ component of vector _____ along the

__________________ of vector _____ and

____________ = ________________________ is the _______________ component of vector _____ along the

__________________ of vector _____.

Example 2: Find the angle $\theta$ between the vectors $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$.

**DEFINITION: ORTHOGONAL VECTORS**

The vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if

Example 3: Determine whether vectors $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ are orthogonal, parallel or neither.

**DIRECTION COSINES**

For a vector in the *plane*, we often measure its direction in terms of the ____________ measured __________________________ from the ______________ ____________ to the ______________.

In *space*, it is more convenient to measure direction in terms of the angles ______________ the nonzero vector $\mathbf{v}$ and the three __________ vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. The angles $\alpha$, $\beta$ and $\gamma$ are the ___________________ ______________ of $\mathbf{v}$ and $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction _______________ of $\mathbf{v}$.

Activity:

1. Use the theorem for the angle between two vectors to find an alternate form of the dot product. Substitute the unit vector $\mathbf{i}$ for vector $\mathbf{u}$.

2. Now find $\mathbf{v} \cdot \mathbf{i}$ using the component form of each vector.

3. Equate your results from parts 1 and 2 and then isolate $\cos\alpha$.

4. Repeat this exercise to find $\cos\beta$ and $\cos\gamma$.

5. Find the normalized form of any nonzero vector $\mathbf{v}$, that is, find two expressions for $\frac{\mathbf{v}}{\|\mathbf{v}\|}$, using your previous results.

6. Find $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$. Hint: $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector.

Example 4: Find the direction angles of the vector $\mathbf{u} = -4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$.

**DEFINITION OF PROJECTION AND VECTOR COMPONENTS**

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors and let $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to $\mathbf{v}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$.

1. $\mathbf{w}_1$ = _______________is called the projection of ____ onto ____ or the vector component of $\mathbf{u}$ _____________ $\mathbf{v}$, and is denoted by ______________________.

2. $\mathbf{w}_2 =$ _____________ is called the vector component of $\mathbf{u}$ _____________________ to $\mathbf{v}$.

**THEOREM: PROJECTION USING THE DOT PRODUCT**

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, then the projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by

**DEFINITION OF WORK**

The work *W* done by a constant force $\mathbf{F}$ as its point of application moves along the vector $\overrightarrow{PQ}$ is given by either of the following:

1. $W =$ ______________________

2. $W =$ ______________________

Example 5: A toy wagon is pulled by exerting a force of 25 pounds on a handle that makes a 20° angle with the horizontal. Find the work done in pulling the wagon 50 feet.

# Section 11.4 The Cross Product of Two Vectors In Space

When you are done with your homework you should be able to…

- π Find the cross product of two vectors in space
- π Use the triple scalar product of three vectors in space

Warm-up: Find the direction cosines of $\mathbf{u} = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and demonstrate that the sum of the squares of the direction cosines is 1.

**DEFINITION OF CROSS PRODUCT OF TWO VECTORS IN SPACE**

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in space.
The cross product of $\mathbf{u}$ and $\mathbf{v}$ is the vector

**THEOREM: ALGEBRAIC PROPERTIES OF THE CROSS PRODUCT**

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in space and let $c$ be a scalar.

1. $\mathbf{u}\times\mathbf{v} =$ ____________
2. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) =$ ____________________
3. $c(\mathbf{u}\times\mathbf{v}) =$ __________ $=$ ___________
4. $\mathbf{u}\times\mathbf{0} =$ _________ $=$ ___
5. $\mathbf{u}\times\mathbf{u} =$ ____
6. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) =$ _________________

   Proof:

**THEOREM: GEOMETRIC PROPERTIES OF THE CROSS PRODUCT**

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in space, and let $\theta$ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

1. $\mathbf{u}\times\mathbf{v}$ is _____________________ to both $\mathbf{u}$ and $\mathbf{v}$.
2. $\|\mathbf{u}\times\mathbf{v}\| =$ ___________________
3. $\mathbf{u}\times\mathbf{v} =$ ___ if and only if $\mathbf{u}$ and $\mathbf{v}$ are _____________ ___________________ of each other.
4. $\|\mathbf{u}\times\mathbf{v}\| =$ the _________ of the ________________________ having $\mathbf{u}$ and $\mathbf{v}$ as ________________ sides.

Example 1: Find $\mathbf{u} \times \mathbf{v}$ and show that it is orthogonal to both $\mathbf{u} = \langle -1,1,2 \rangle$ and $\mathbf{v} = \langle 0,1,0 \rangle$.

In physics, the cross product can be used to measure ____________, which is the _________________ $\mathbf{M}$ of a ___________ $\mathbf{F}$ about a point *P*. If the point of application of the force is *Q*, the moment of $\mathbf{F}$ about *P* is given by ____________________. The magnitude of the moment $\mathbf{M}$ measures the ________________ of the vector $\overrightarrow{PQ}$ to ____________ counterclockwise about an _________ directed along the vector______.

Example 2: Both the magnitude and direction of the force on a crankshaft change as the crankshaft rotates. Find the torque on the crankshaft using the position and data shown in the figure.

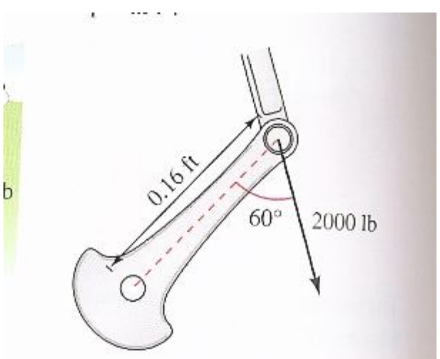


Figure for 38

**THEOREM: THE TRIPLE SCALAR PRODUCT**

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$,
The triple scalar product is given by

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) =$$

Note: The volume of a ________________________________ with vectors ______, ______, and _______ as adjacent edges is given by _________________________.

Example 3: Find $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. $\mathbf{u} = \langle 1,1,1 \rangle$, $\mathbf{v} = \langle 2,1,0 \rangle$, $\mathbf{w} = \langle 0,0,1 \rangle$.

Example 4: Find the volume of the parallelepiped having adjacent edges $\mathbf{u} = \langle 1,3,1 \rangle$, $\mathbf{v} = \langle 0,6,6 \rangle$, $\mathbf{w} = \langle -4,0,-4 \rangle$.

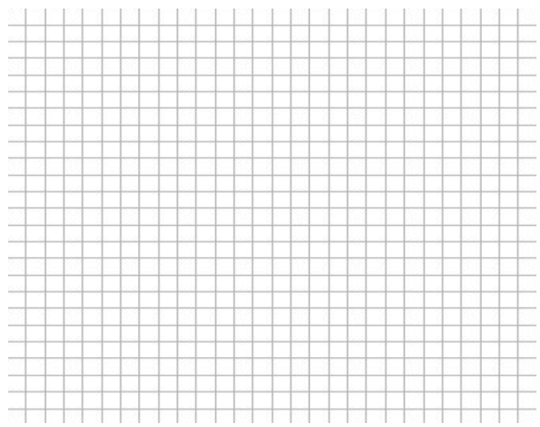
# Section 11.5 Lines and Planes In Space

When you are done with your homework you should be able to…

- π Write a set of parametric equations for a line in space
- π Write a linear equation to represent a plane in space
- π Sketch the plane given by a linear equation
- π Find the distance between points, planes, and lines in space

Warm-up: Graph the following parametric curve, indicating the orientation.

$x-3=\cos^2\theta,\ \text{and}\ y=\sin^2\theta,\ 0\le\theta<2\pi$



In the plane _______________ is used to determine an equation of a line. In space, it is convenient to use

_________________ to determine the equation of a line.

**THEOREM: PARAMETRIC EQUATIONS OF A LINE IN SPACE**

A line *L* parallel to the vector ____________ and passing through the point ___________ is represented by the **parametric equations**

Note: If the direction numbers *a, b,* and *c* are all _______________, you can eliminate the parameter *t* to obtain __________________ _________________ of the line.

Example 1: Find equations of the line which passes through the point $\left(-3,0,2\right)$ and is parallel to the vector $\mathbf{v}=-2\mathbf{i}+8\mathbf{j}-3\mathbf{k}$ in:

a. Parametric form

b. Symmetric form

**THEOREM: STANDARD EQUATION OF A PLANE IN SPACE**

The plane containing the point $(x_1, y_1, z_1)$ and having normal vector $\mathbf{n} = \langle a, b, c \rangle$ can be represented, in **standard form**, by the equation

The **general form** is given by the equation

**THEOREM: DISTANCE BETWEEN A POINT AND A PLANE**

The distance between a plane and a point *Q* (not in the plane) is

where *P* is a point in the plane and $\mathbf{n}$ is normal to the plane. Other forms of

this distance from a point $Q(x_0, y_0, z_0)$ and the plane given by $ax + by + cz + d = 0$ are as follows:

or

Example 2: Find an equation of the plane passing through the point $(1,0,-3)$ perpendicular to the vector $\mathbf{n}=\mathbf{k}$.

**THEOREM: DISTANCE BETWEEN A POINT AND A LINE IN SPACE**

The distance between a point $Q$ and a line in space is given b

where $\mathbf{u}$ is a direction vector for the line and $P$ is a point on the line.

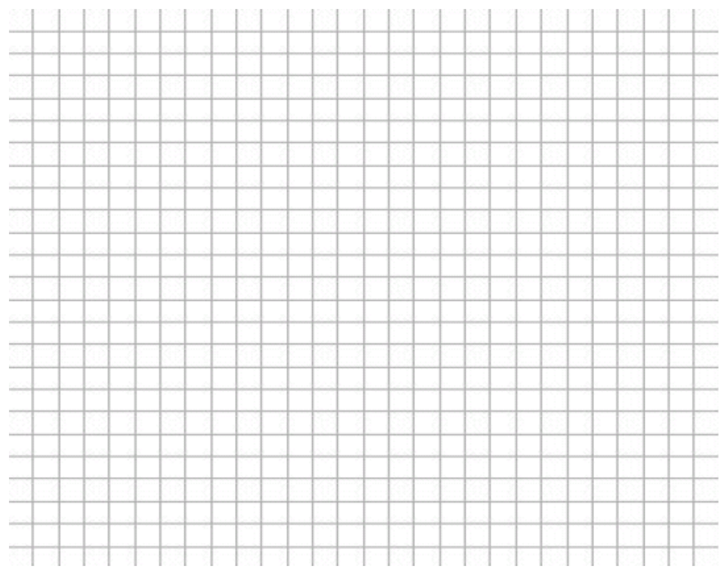
Example 3: Find the distance between the point $(3,2,1)$ and the plane $x-y+2z=4$.

# Section 11.6 Surfaces In Space

When you are done with your homework you should be able to…

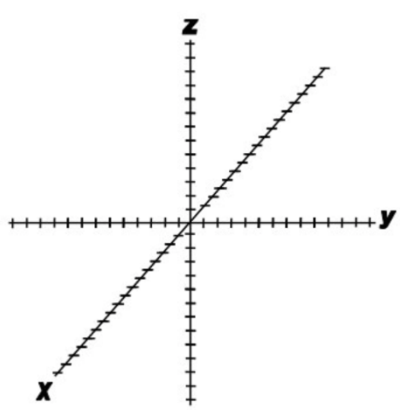
- π Recognize and write equations for cylindrical surfaces
- π Recognize and write equations for quadric surfaces
- π Recognize and write equations for surfaces of revolution

Warm-up: Find the volume of the region bounded by the graphs $y=4,\ x=4,\ x=0,$ and $y=0$ which has been rotated about the x-axis. Graph the resulting solid.



**DEFINITION OF A CYLINDER**

Let *C* be a curve in a plane and let *L* be a line not in a parallel plane. The set of all lines parallel to *L* and intersecting *C* is called a **cylinder**. *C* is called the **generating curve (aka directrix)** of the cylinder and the parallel lines are called **rulings**.

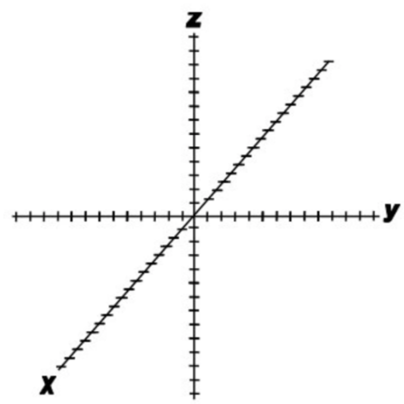


## EQUATIONS OF CYLINDERS

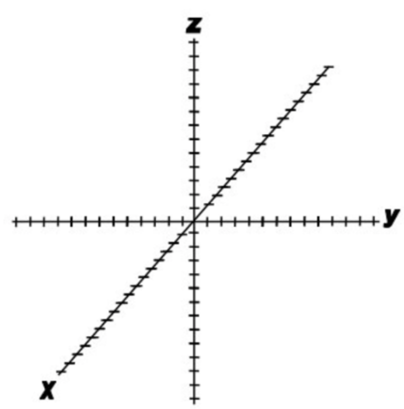
The equation of a cylinder whose rulings are parallel to one of the coordinate axes contains only the variables corresponding to the other two axes.

Example 1: Sketch the surface represented by each equation.

$y = z^2$




$z = \cos x$




## QUADRIC SURFACE

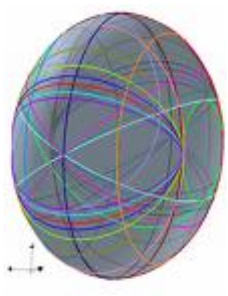
The equation of a **quadric surface** in space is a second-degree equation of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

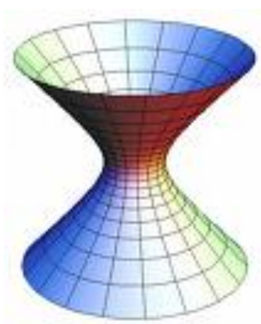
There are six basic types of quadric surfaces:

**Ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic cone, elliptic paraboloid, and hyperbolic paraboloid.**

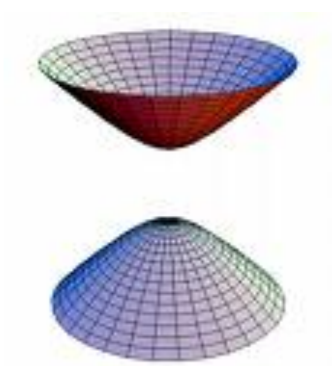
Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Trace Plane



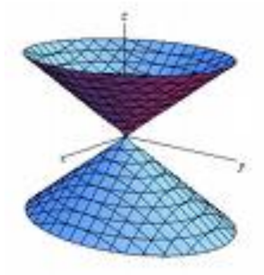
Hyperboloid (1 sheet) $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Trace Plane



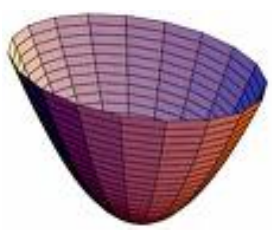
Hyperboloid (2 sheets) $\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ Trace Plane



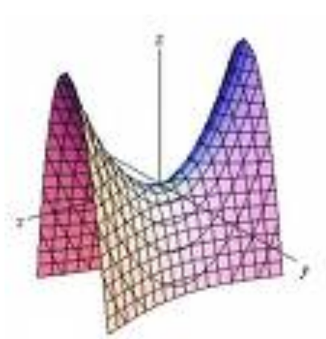
Elliptic Cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ Trace Plane



Elliptic Paraboloid $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Trace Plane

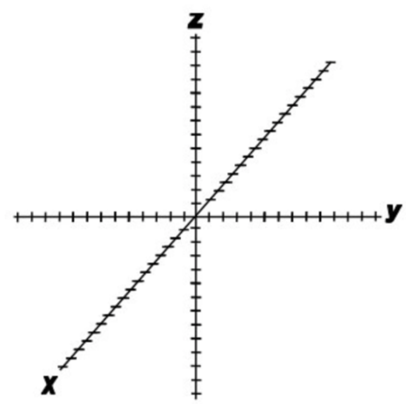


Hyperbolic Paraboloid $z=\frac{y^2}{b^2}-\frac{x^2}{a^2}$ Trace Plane



Example 2: Identify and sketch the quadric surface.

$$\frac{x^2}{16}+\frac{y^2}{25}+\frac{z^2}{25}=1$$




**SURFACE OF REVOLUTION**

If the graph of a radius function *r* is revolved about one of the coordinate axes, the equation of the resulting surface of revolution has one of the following forms:

Revolved about the x-axis: $y^2+z^2=\left[r(x)\right]^2$

Revolved about the y-axis: $x^2+z^2=\left[r(y)\right]^2$

Revolved about the z-axis: $x^2+y^2=\left[r(z)\right]^2$

Example 3: Find an equation for the surface of revolution generated by revolving the curve $z=3y$ in the yz-plane about the y-axis.

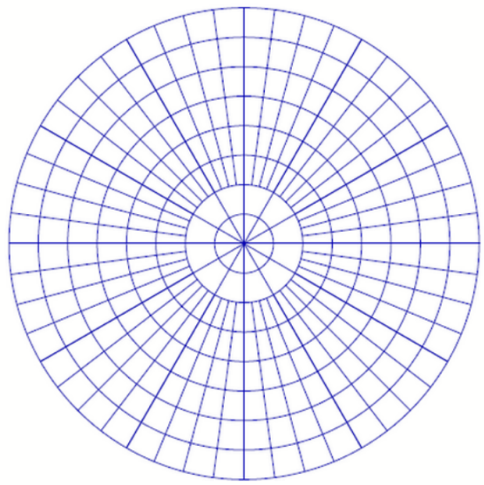
# Section 11.7 Cylindrical and Spherical Coordinates

When you are done with your homework you should be able to…

- π Use cylindrical coordinates to represent surfaces in space
- π Use spherical coordinates to represent surfaces in space

Warm-up: Convert the rectangular equation to polar form and sketch its graph by hand.

$y^2 = 9x$



**THE CYLINDRICAL COORDINATE SYSTEM**

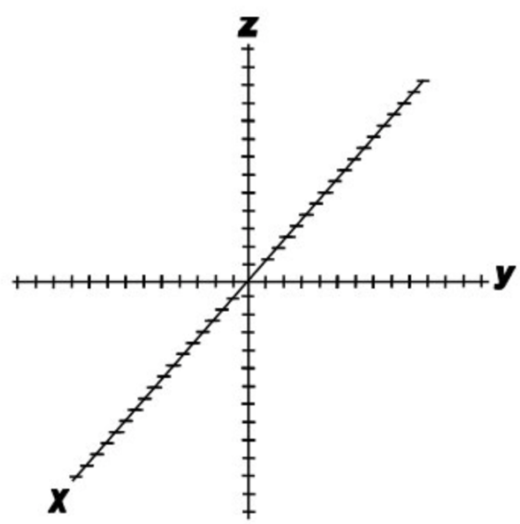
In a **cylindrical coordinate system** a point *P* in space is represented by an ordered triple ____________.

1. $(r,\theta)$ is a polar representation of the projection of *P* in the *xy*-plane.
2. $z$ is the directed distance from $(r,\theta)$ to *P*.

**Conversion Guidelines**

*Cylindrical to rectangular:* $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

*Rectangular to cylindrical:* $r^2 = x^2 + y^2,\ \tan\theta = \dfrac{y}{x},\ z = z$



Example 1: Convert the point $\left(-2,\frac{2\pi}{3},5\right)$ to rectangular coordinates.

Example 2: Convert the point $\left(3,\sqrt{3},-1\right)$ to cylindrical coordinates.

Example 3: Find an equation in cylindrical coordinates for the equation $x^2+y^2=8x$, given in rectangular coordinates.

**THE SPHERICAL COORDINATE SYSTEM**

In a **spherical coordinate system**, a point *P* in space is represented by an ordered triple $(\rho,\theta,\phi)$.

$\rho$ is the __________________ between *P* and the _________________, $\rho \geq 0$.

$\theta$ is the same angle used in cylindrical coordinates for __________.

$\phi$ is the angle _______________ the positive *z*-axis and the line segment $\overline{OP}$, $0 \leq \phi \leq \pi$.

Note that the first and third coordinates, $\rho$ and $\phi$, are nonnegative. $\rho$ is the lowercase Greek letter *rho* and $\phi$ is the lowercase Greek letter *phi*.

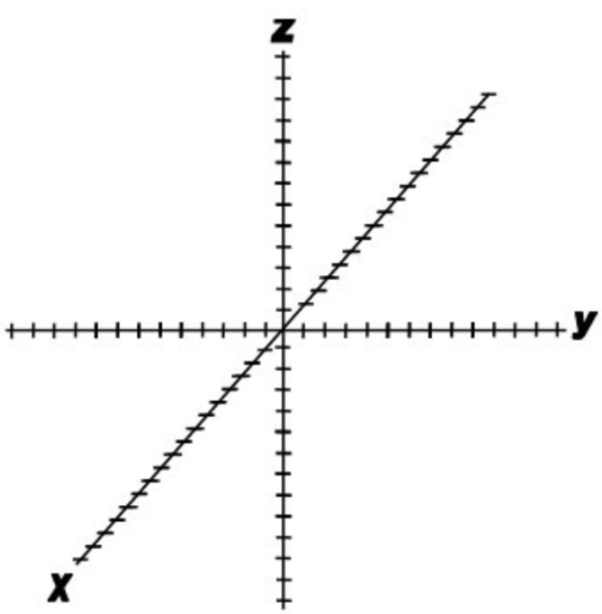
**Conversion Guidelines**

*Spherical to rectangular:* $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$

*Rectangular to spherical:* $\rho^2 = x^2 + y^2 + z^2,\ \tan\theta = \dfrac{y}{x},\ \phi = \arccos\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right)$

*Spherical to cylindrical* $r \geq 0$: $r^2 = \rho^2\sin^2\phi,\ \theta = \theta,\ z = \rho\cos\phi$

*Cylindrical to spherical* $r \geq 0$: $\rho = \sqrt{r^2+z^2},\ \theta = \theta,\ \phi = \arccos\left(\dfrac{z}{\sqrt{r^2+z^2}}\right)$



Example 4: Convert the point given in cylindrical coordinates $\left(3,-\frac{\pi}{4},0\right)$ to spherical coordinates.

Example 5: Find an equation in spherical coordinates for the equation $x^2+y^2-3z^2=0$, given in rectangular coordinates.

# Chapter 12 Vector Valued Functions

## Section 12.1 Vector-Valued Functions

When you are done with your homework you should be able to…

- π Analyze and sketch a space curve given by a vector-valued function
- π Extend the concepts of limits and continuity to vector-valued functions

Warm-up: Evaluate the following limits analytically.

1. $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}$

2. $\lim\limits_{t\to 4}\dfrac{t^2-16}{t^2-4t}$

3. $\lim\limits_{x\to\infty}\left(e^{-x}-\dfrac{6}{x}-\arctan x\right)$

**DEFINITION OF VECTOR-VALUED FUNCTION**

A function of the form

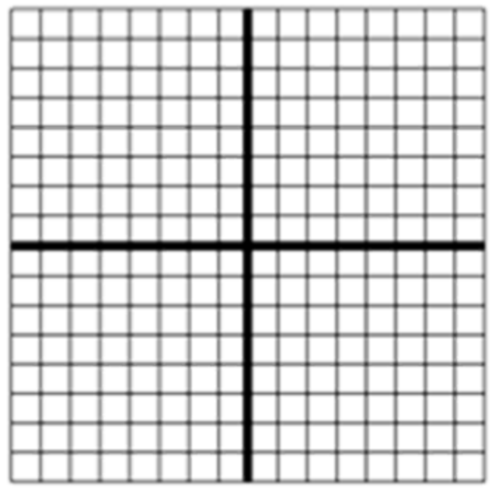
is a **vector-valued function**, where the **component functions** *f, g,* and *h* are real-valued functions of the parameter *t*. The domain is considered to be the intersection of the domains of the component functions *f, g,* and *h,* unless stated otherwise.

Example 1: Find the domain of the vector-valued function.

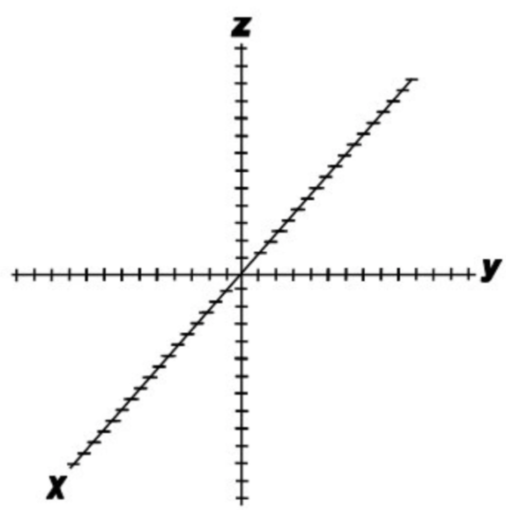
$$\mathbf{r}(t) = \sqrt{4-t^2}\,\mathbf{i} + t^2\mathbf{j} - 6t\mathbf{k}$$

Example 2: Sketch the curve represented by the vector-valued function.

a. $\mathbf{r}(t) = (1-t)\mathbf{i} + \sqrt{t}\,\mathbf{j}$



b. $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + \frac{t}{2}\mathbf{k}$



**DEFINITION OF THE LIMIT OF A VECTOR-VALUED FUNCTION**

If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then

provided *f* and *g* have limits as $t \to a$.

If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

provided *f, g* and *h* have limits as $t \to a$.

**DEFINITION OF CONTINUITY OF A VECTOR-VALUED FUNCTION**

A vector-valued function $\mathbf{r}$ is **continuous at the point** given by $t = a$ if the limit of $\mathbf{r}(t)$ exists as $t \to a$ and $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$.

A vector-valued function $\mathbf{r}$ is **continuous on an interval** *I* if it is continuous at every point in the interval.

Example 3: Evaluate the limit and determine the interval(s) on which the vector-valued function is continuous.

$$\lim_{t \to 1} \left( (\ln t)\mathbf{i} - \left( \frac{1 - t^2}{1 - t} \right)\mathbf{j} + (\arcsin t)\mathbf{k} \right)$$

# Section 12.2 Differentiation and Integration of Vector-Valued Functions

When you are done with your homework you should be able to…

- π Differentiate a vector-valued function
- π Integrate a vector-valued function

Warm-up 1: Evaluate the following derivatives with respect to *x*.

1. $y=\dfrac{\sin^2 3x}{\sqrt{x}}$

2. $f\left(x\right)=xe^{-2x}$

3. $y=\ln\left(\dfrac{5x}{e^{x^2}}\right)^{2/3}-\dfrac{6}{x}-\arctan 3x^3$

Warm-up 2: Integrate.

1. $\int \left(6x^2 - \sin^2 3x\right) dx$

2. $\int \frac{\sqrt{\ln x}}{x}\, dx$

3. $\int \frac{4}{\sqrt{1-x^2}}\, dx$

**DEFINITION OF THE DERIVATIVE OF A VECTOR-VALUED FUNCTION**

The **<u>derivative of a vector-valued function $\mathbf{r}$</u>** is defined by

for all *t* for which the limit exists. If $\mathbf{r}'(c)$ exists, then $\mathbf{r}$ is **<u>differentiable at *c*</u>**. If $\mathbf{r}'(c)$ exists for all *c* in an open interval *I* then $\mathbf{r}$ is **<u>differentiable on the open interval *I*</u>**. Differentiability of vector-valued functions can be extended to closed intervals by considering one-sided limits.

Other notation:

**THEOREM: DIFFERENTIATION OF VECTOR-VALUED FUNCTIONS**

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where $f$ and $g$ are differentiable functions of *t*, then ________________________________. provided *f* and *g* have limits as $t \to a$.

2. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where $f$, $g$, and $h$ are differentiable functions of *t*, then ________________________________________.

**<u>Higher-order derivatives</u>** of vector-valued functions are obtained by successive differentiation of each component function.

The ____________________________ of the _____________________ represented by the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is **<u>smooth on an open interval *I*</u>** if $f'$, $g'$, and $h'$ are continuous on *I* and $\mathbf{r}'(t) \neq \mathbf{0}$ for any value of *t* on the open interval *I*.

**THEOREM: PROPERTIES OF THE DERIVATIVE**

Let $\mathbf{r}$ and $\mathbf{u}$ be differentiable vector-valued functions of *t,* let $f$ be a differentiable real-valued function of *t,* and let *c* be a scalar.

1. $D_t\left[c\mathbf{r}(t)\right]=c\mathbf{r}'(t)$

2. $D_t\left[\mathbf{r}(t)\pm\mathbf{u}(t)\right]=\mathbf{r}'(t)\pm\mathbf{u}'(t)$

3. $D_t\left[f(t)\mathbf{u}(t)\right]=f(t)\mathbf{r}'(t)+f'(t)\mathbf{r}(t)$

4. $D_t\left[\mathbf{r}(t)\cdot\mathbf{u}(t)\right]=\mathbf{r}(t)\cdot\mathbf{u}'(t)+\mathbf{r}'(t)\cdot\mathbf{u}(t)$

5. $D_t\left[\mathbf{r}(t)\times\mathbf{u}(t)\right]=\mathbf{r}(t)\times\mathbf{u}'(t)+\mathbf{r}'(t)\times\mathbf{u}(t)$

6. $D_t\left[\mathbf{r}(f(t))\right]=\mathbf{r}'(f(t))\cdot f'(t)$

7. If $\mathbf{r}(t)\cdot\mathbf{r}(t)=c$, then $\mathbf{r}(t)\cdot\mathbf{r}'(t)=0$

Example 1: Find $\mathbf{r}'(t)\cdot\mathbf{r}''(t)$.

$\mathbf{r}(t)=\left(t^2+t\right)\mathbf{i}+\left(t^2-t\right)\mathbf{j}$

Example 2: Find $D_t\left[\mathbf{r}(t)\times\mathbf{u}(t)\right]$

$\mathbf{r}(t)=t\mathbf{i}+2\sin t\mathbf{j}+2\cos t\mathbf{k},$

$\mathbf{u}(t)=\frac{1}{t}\mathbf{i}+2\sin t\mathbf{j}+2\cos t\mathbf{k},$

**DEFINITION OF INTEGRATION OF VECTOR-VALUED FUNCTIONS**

If $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}$, where $f$ and $g$ are continuous on $[a,b]$ then the **indefinite integral (antiderivative)** of $\mathbf{r}$ is

and its **definite integral** over the interval $a\le t\le b$ is

If $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$, where $f$, $g$, and $h$ are continuous on $[a,b]$ then the **indefinite integral (antiderivative)** of $\mathbf{r}$ is

and its **definite integral** over the interval $a\le t\le b$ is

Example 3: Evaluate the indefinite integral

$$\int \left( 4t^3\mathbf{i} + 6t\mathbf{j} - 4\sqrt{t}\mathbf{k} \right) dt$$

Example 4: Evaluate the definite integral

$$\int_0^{\pi/4} \left[ \sec t \tan t\mathbf{i} + \tan t\mathbf{j} + 2\sin t \cos t\mathbf{k} \right] dt$$

# Section 12.3 Velocity and Acceleration

When you are done with your homework you should be able to…

- π Describe the velocity and acceleration associated with a vector-valued function
- π Use a vector-valued function to analyze projectile motion

Warm-up: Consider the circle given by $\mathbf{r}(t) = (\cos \omega t)\mathbf{i} + (\sin \omega t)\mathbf{j}$. Use a graphing calculator in parametric mode to graph this circle for several values of $\omega$.

How does $\omega$ affect the velocity of the terminal point as it traces out the curve?

For a given value of $\omega$, does the speed appear constant?

Does the acceleration appear constant?

**DEFINITIONS OF VELOCITY AND ACCELERATION**

If x and y are twice differentiable functions of t, and $\mathbf{r}$ is a vector-valued function given by $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}$, then the velocity vector, acceleration vector, and speed at time t are as follows:

**Velocity:**

**Acceleration:**

**Speed:**

For motion along a space curve, the definitions are as follows:

**Velocity:**

**Acceleration:**

**Speed:**

Example 1: The position vector $\mathbf{r}(t)=3\cos t\mathbf{i}+2\sin t\mathbf{j}$ describes the path of an object moving in the xy-plane. Sketch a graph of the path and sketch the velocity and acceleration vectors at the point $(3,0)$.

Example 2: The position vector $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + 2t^{3/2}\mathbf{k}$ describes the path of an object moving in space. Find the velocity, speed and acceleration of the object.

**THEOREM: POSITION FUNCTION FOR A PROJECTILE**

Neglecting air resistance, the path of a projectile launched from an initial height *h* with initial speed $v_0$ and angle of elevation $\theta$ is described by the vector function

where ______ is the __________________________ constant.

Example 3: Determine the maximum height and range of a projectile fired at a height 3 feet above the ground with an initial velocity of 900 feet per second and at an angle of $45^\circ$ above the horizontal.

Example 4: A baseball is hit from a height of 2.5 feet above the ground with an initial velocity of 140 feet per second and at an angle of 22˚ above the horizontal. Use a graphing utility to graph the path of the ball and determine whether it will clear a ten-foot-high fence located 375 feet from home plate.

Example 5: Find the maximum speed of a point on the circumference of an automobile tire of radius one foot when the automobile is traveling at 55 mph. Compare this speed with the speed of the automobile. Use the following formula for the cycloid:

$$\mathbf{r}(t) = b(\omega t - \sin\omega t)\mathbf{i} + b(1 - \cos\omega t)\mathbf{j}$$

$\omega$ is the constant angular velocity of the circle and $b$ is the radius of the circle.

# Section 12.4 Tangent Vectors and Normal Vectors

When you are done with your homework you should be able to…

- π Find a unit tangent vector at a point on a space curve
- π Find the tangential and normal components of acceleration

Warm-up: Consider the two curves given by $y_1 = 1 - x^2$ and $y_2 = x^2 - 1$.

a. Find the unit tangent vectors to each curve at their points of intersection.

b. Find the angles ( $0 \le \theta \le 90^\circ$ ) between the curves at their points of intersection.

**DEFINITION OF UNIT TANGENT VECTOR**

Let *C* be a smooth curve represented by $\mathbf{r}$ on an open interval *I*. The **unit tangent vector** $\mathbf{T}(t)$ at $t$ is defined to be

The **tangent line to a curve** at a point is the line passing through point and parallel to the unit tangent vector.

Example 1: Find the unit tangent vector to the curve $\mathbf{r}(t) = e^t \cos t\mathbf{i} + e^t \mathbf{j}$ when $t = 0$.

Example 2: Consider the space curve $\mathbf{r}(t)=\left\langle t,t,\sqrt{4-t^2}\right\rangle$ at the point $\left(1,1,\sqrt{3}\right)$.

a. Find the unit tangent vector at the given point.

b. Find a set of parametric equations for the line tangent to the space curve at the given point.

**DEFINITION: PRINCIPAL UNIT NORMAL VECTOR**

Let *C* be a smooth curve represented by $\mathbf{r}$ on an open interval *I*. If $\mathbf{T}'(t) \neq \mathbf{0}$, then the **principal unit normal vector** $\mathbf{N}(t)$ at $t$ is defined to be

At any point on a curve, a unit normal vector is ______________________ to the unit tangent vector. The principal unit normal vector points in the direction in which the curve is turning.

Example 3: Find the principal unit normal vector to the curve $\mathbf{r}(t) = \ln t\mathbf{i} + (t+1)\mathbf{j}$ at the time $t = 2$.

**THEOREM: ACCELERATION VECTOR**

If $\mathbf{r}(t)$ is the position vector for a smooth curve *C* and $\mathbf{N}(t)$ exists, then the acceleration vector

lies in the plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

**THEOREM: TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION**

If $\mathbf{r}(t)$ is the position vector for a smooth curve *C* and $\mathbf{N}(t)$ exists, then the **tangential and normal components of acceleration are as follows**:

Note that $a_{\mathbf{N}} \geq 0$. The normal component of acceleration is also called the **centripetal component of acceleration.**

Example 4: Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$ for the plane curve $\mathbf{r}(t)=e^{t}\mathbf{i}+e^{-t}\mathbf{j}+t\mathbf{k}$ at the time $t=0$.

# Section 12.5 Arc Length and Curvature

When you are done with your homework you should be able to…

- π Find the arc length of a space curve
- π Use the arc length parameter to describe a plane curve or space curve
- π Find the curvature of a curve at a point on the curve
- π Use a vector-valued function to find frictional force

Warm-up: Find the arc length of the curve $x = \arcsin t$ and $y = \ln\sqrt{1-t^2}$ on the interval $\left[0, \frac{1}{2}\right]$.

**THEOREM: ARC LENGTH OF A SPACE CURVE**

If *C* is a smooth curve given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ on an interval $[a, b]$, then the **arc length of *C*** is

Example 1: Find the arc length of the curve given by $\mathbf{r}(t) = 2\sin t\mathbf{i} + 5t\mathbf{j} + 2\cos t\mathbf{k}$ over $[0, \pi]$.

**DEFINITION: ARC LENGTH FUNCTION**

Let *C* be a smooth curve given by $\mathbf{r}(t)$ defined on the closed interval $[a,b]$, then the **<u>arc length of *C*</u>** is

The arc length *s* is called the **<u>arc length parameter</u>**. The arc length function is nonnegative as it measures the distance along *C* from the initial point. Using the definition of the arc length function and the second fundamental theorem of calculus, you can conclude____________________.

Note: For _________________ along a curve, the convenient ____________________ is time _____. For studying _____________________ properties of a curve, the convenient parameter is the ________ length _____.

Example 2: Find the arc length function for the line segment given by $\mathbf{r}(t) = (3-3t)\mathbf{i} + 4t\mathbf{j},\ 0 \le t \le 1$.and write $\mathbf{r}$ as a function of the parameter *s.*

**THEOREM: ARC LENGTH PARAMETER**

If *C* is a smooth curve given by $\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$ or $\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k}$ where *s* is the **arc length parameter**, then

Moreover, if *t* is any parameter for the vector-valued function $\mathbf{r}$ such that $\left\|\mathbf{r}'(s)\right\| = 1$, then _____ must be the arc length parameter.

**DEFINITION OF CURVATURE**

Let *C* be a smooth curve (in the plane or in space) given by $\mathbf{r}(s)$,where *s* is the arc length parameter. The **curvature** *K* at *s* is given by

Example 3: Find the curvature *K* of the curve, where *s* is the arc length parameter.

$\mathbf{r}(s) = (3+s)\mathbf{i} + \mathbf{j}$

**THEOREM: FORMULAS FOR CURVATURE**

If $C$ is a smooth curve given by $\mathbf{r}(t)$, then the curvature $K$ of $C$ at $t$ is given by

Example 4: Find the curvature $K$ of the curve $\mathbf{r}(t) = 2t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$.

**THEOREM: CURVATURE IN RECTANGULAR COORDINATES**

If $C$ is the graph of a twice differentiable function given by $y = f(x)$, then the curvature $K$ at the point $(x, y)$ is given by

Related Stuff: Let $C$ be a curve with curvature $K$ at point $P$. The circle passing through point $P$ with radius ___________ is called the ___________________**of curvature** if the circle lies on the concave side of the curve and shares a common tangent line with the curve at point $P$. The radius is called the **radius of curvature** at $P$ and the center of the circle is called the **center of curvature**.

Example 5: Find the curvature and radius of curvature of the plane curve $y = 2x + \frac{4}{x}$ at $x = 1$.

**THEOREM: ACCELERATION, SPEED, AND CURVATURE**

If $\mathbf{r}(t)$ is the position vector for a smooth curve *C* then the acceleration vector is given by

where *K* is the curvature of *C* and $\frac{ds}{dt}$ is the speed.

**Frictional Force**

A moving object with mass *m* is in contact with a stationary object. The total force required to produce an acceleration $\mathbf{a}$ along a given path is

Example 6: A 6400-pound vehicle is driven at a speed of 35 mph on a circular interchange of radius 250 feet. To keep the vehicle from skidding off course, what frictional force must the road surface exert on the tires?

# Chapter 13 Functions of Several Variables

## Section 13.1 Introduction to Functions of Several Variables

When you are done with your homework you should be able to…

- $\pi$ Understand the notation for a function of several variables
- $\pi$ Sketch the graph of a function of two variables
- $\pi$ Sketch level curves for a function of two variables
- $\pi$ Sketch level surfaces for a function of three variables

Warm-up: Find two functions such that the composition $h(x)=(f\circ g)(x)=\sin^2 x$

$f(x)=$ _______________

$g(x)=$ _______________

**DEFINITION: A FUNCTION OF TWO VARIABLES**

Let *D* be a set of ordered pairs of real numbers. If to each ordered pair $(x,y)$ in *D* there corresponds a unique real number $f(x,y)$, then $f$ is called a ________________ of ____ and ____. The set *D* is the _____________ of ____, and the corresponding set of values for $f(x,y)$ is the _________ of *f*.

Example 1: Find and simplify the function values.

$$g(x,y) = \ln|x+y|$$

a. $g(2,3)$ b. $g(e,0)$ c. $g(0,1)$

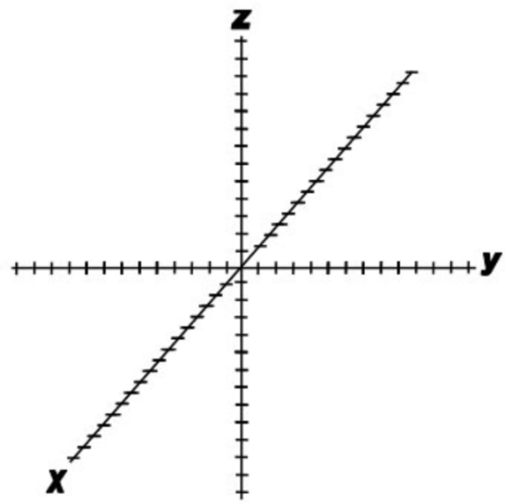
Example 2: Describe the domain and range of each function.

a. $f(x,y) = \arccos\left(\frac{y}{x}\right)$

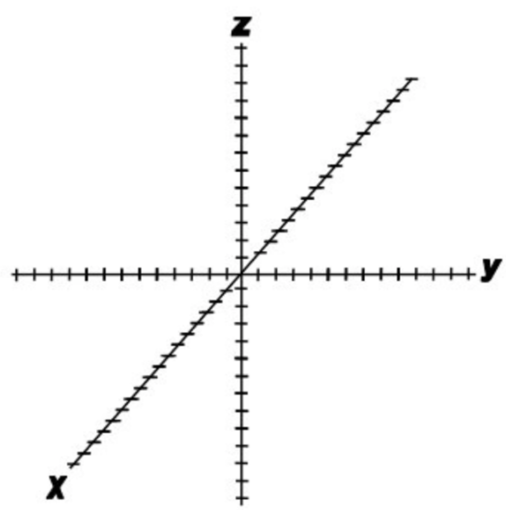
b. $g(x,y) = x\sqrt{y}$

Example 3: Sketch the surface given by the function.

a. $g(x,y) = \left(\frac{1}{2}\right)x$




b. $z = \frac{1}{2}\sqrt{x^2 + y^2}$

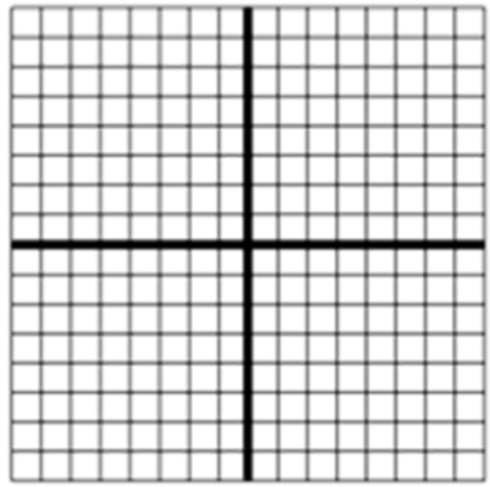


**LEVEL CURVES**

We can also visualize a function of two variables using a ________________ ___________. This involves assigning a ______________ value to *z.* This is then assigned to the point (*x, y*).

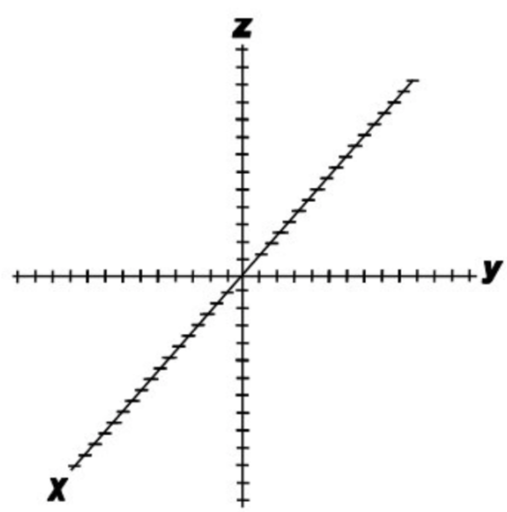
Example 4: Describe the level curves of the function. Sketch the level curves for the given *c*-values.

$$f(x,y)=\frac{x}{x^2+y^2},\quad c=\pm\frac{1}{2},\ \pm1,\ \pm\frac{3}{2},\ \pm2$$



Example 5: Sketch the graph of the level surface $f(x,y,z)=c$ at the given value of *c*.

$$f(x,y,z)=\sin x-z,\quad c=0$$



# Section 13.2 Limits and Continuity

When you are done with your homework you should be able to…

- π Understand the definition of a neighborhood in the plane
- π Understand and use the definition of the limit of a function of two variables
- π Extend the concept of continuity to a function of two variables
- π Extend the concept of continuity to a function of three variables

Warm-up: Evaluate the following limits analytically.

a. $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}$

b. $\lim\limits_{t\to 4}\dfrac{t^2-16}{t^2-4t}$

c. $\lim\limits_{x\to\infty}\left(e^{-x}-\dfrac{6}{x}-\arctan x\right)$

**DEFINITION OF VECTOR-VALUED FUNCTION**

A function of the form

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} = \langle f(t), g(t) \rangle \quad \text{plane}$$

or

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} = \langle f(t), g(t), h(t) \rangle \quad \text{space}$$

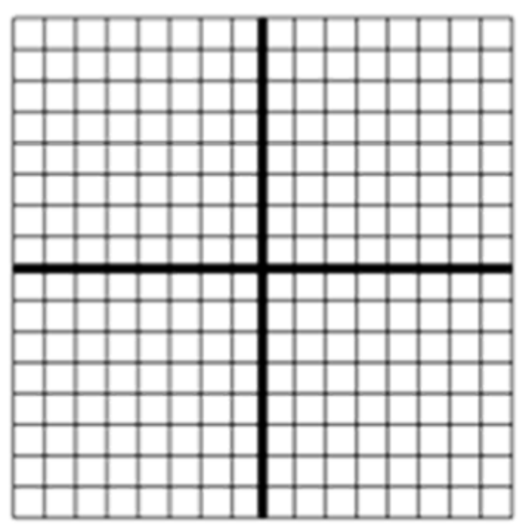
is a ________________-______________ function, where the _______________ functions *f, g,* and *h* are ___________-_______________ functions of the parameter *t*. The domain is considered to be the ___________________ of the domains of the component functions *f, g,* and *h,* unless stated otherwise.

Example 1: Find the domain of the vector-valued function.

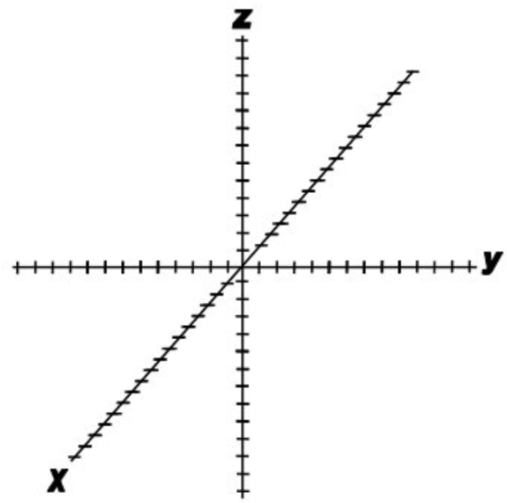
$$\mathbf{r}(t) = \sqrt{4 - t^2}\,\mathbf{i} + t^2\mathbf{j} - 6t\mathbf{k}$$

Example 2: Sketch the curve represented by the vector-valued function.

a. $\mathbf{r}(t) = (1 - t)\mathbf{i} + \sqrt{t}\,\mathbf{j}$



b. $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + \frac{t}{2}\mathbf{k}$




**DEFINITION OF THE LIMIT OF A VECTOR-VALUED FUNCTION**

If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j}$$

provided *f* and *g* have limits as $t \to a$.

If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} + \left[\lim_{t\to a} h(t)\right]\mathbf{k}$$

provided *f, g* and *h* have limits as $t \to a$.

**DEFINITION OF CONTINUITY OF A VECTOR-VALUED FUNCTION**

A vector-valued function $\mathbf{r}$ is **continuous at the point** given by $t = a$ if the limit of $\mathbf{r}(t)$ exists as $t \to a$ and $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$.

A vector-valued function $\mathbf{r}$ is **continuous on an interval** *I* if it is continuous at every _____________ in the interval.

Example 3: Evaluate the limit and determine the interval(s) on which the vector-valued function is continuous.

$$\lim_{t\to 1}\left( (\ln t)\mathbf{i} - \left(\frac{1-t^2}{1-t}\right)\mathbf{j} + (\arcsin t)\mathbf{k} \right)$$

# Section 13.3 Partial Derivatives

When you are done with your homework you should be able to…

- π Find and use partial derivatives of a function of two variables
- π Find and use partial derivatives of a function of three or more variables
- π Find higher-order partial derivatives of a function of two or three variables

Warm-up: Find the derivative of the following functions. Simplify your result to a single rational expression with positive exponents.

a. $f(x)=\dfrac{3x^2-x+2}{\sqrt{x}}$

b. $g(x)=(5x-3)^2$

c. $f(x)=\cos\left(x-\dfrac{\pi}{4}\right)$

**DEFINITION: PARTIAL DERIVATIVES OF A FUNCTION OF TWO VARIABLES**

If $z = f(x, y)$ then the **first partial derivatives** of $f$ with respect to $x$ and $y$ are $f_x$ and $f_y$ defined by

provided the limit exists.

Example 1: Find the partial derivatives $f_x$ and $f_y$ of the following functions.

a. $f(x, y) = x^2 - 2y^2 + 4$

b. $z = \sin 5x \cos 5y$

c. $f(x,y)=\int_x^y (2t+1)\,dt+\int_y^x (2t-1)\,dt$

**NOTATION FOR FIRST PARTIAL DERIVATIVES FOR** $z=f(x,y)$

Example 2: Use the limit definition to find the first partial derivatives with respect to $x$, $y$ and $z$.

$$f(x,y,z) = 3x^2y - 5xyz + 10yz^2$$

**PARTIAL DERIVATIVES OF A FUNCTION OF THREE OR MORE VARIABLES**

If $w = f(x, y, z)$ then the **first partial derivatives** of $f$ with respect to $x$, $y$ and $z$ are defined by

$$\frac{dw}{dx} = f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$
$$\frac{dw}{dy} = f_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$
$$\frac{dw}{dz} = f_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

provided the limit exists.

Example 3: Find $f_x, f_y \text{ and } f_z$ at the given point.

$$f(x, y, z) = \frac{xy}{x + y + z}, \quad (3, 1, -1)$$

**HIGHER ORDER PARTIAL DERIVATIVES**

1. Differentiate twice with respect to *x.*

2. Differentiate twice with respect to *y.*

3. Differentiate first with respect to *x* and then with respect to *y.*

4. Differentiate first with respect to *y* and then with respect to *x.*

Example 4: Find the four second partial derivatives.

a. $z = \ln\left(x - y\right)$

b. $z = \arctan\left(\frac{y}{x}\right)$

**THEOREM: EQUALITY OF MIXED PARTIAL DERIVATIVES**

If $f$ is a function of $x$ and $y$ such that $f_{xy}$ and $f_{yx}$ are continuous on an open disk $R$, then, for every $(x, y)$ in $R$,

Example 5: Find the slopes of the surface in the *x*- and *y*-directions at the given point.

$h(x, y) = x^2 - y^2, \quad (-2, 1, 3)$

# Section 13.4 Differentials

When you are done with your homework you should be able to…

- π Understand the concepts of increments and differentials
- π Extend the concept of differentiability to a function of two variables
- π Use a differential as an approximation

Warm-up: The measurement of a side of a square is found to be 12 inches, with a possible error of $\frac{1}{64}$ inch. Use differentials to approximate the possible propagated error in computing the area of the square.

**DEFINITION OF TOTAL DIFFERENTIAL**

If $z = f(x, y)$ and $\Delta x$ and $\Delta y$ are increments of $x$ and $y$, then the **differentials** of the independent variables $x$ and $y$ are

and the **total differential** of the dependent variable $z$ is

Example 1: Find the total differential.

a. $z = \dfrac{x^2}{y}$

b. $w = e^y \cos x + z^2$

**DEFINITION OF DIFFERENTIABILITY**

A function $f$ given by $z = f(x,y)$ is <u>differentiable</u> at $(x_0, y_0)$ if $\Delta z$ can be written in the form

where both $\varepsilon_1$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. The function $f$ is <u>**differentiable in a region *R***</u> if it is differentiable at each point in *R.*

Example 2: Find $z = f(x,y)$ and use the total differential to approximate the quantity.

$$(2.03)^2(1+8.9)^3 - 2^2(1+9)^3$$

**THEOREM: SUFFICIENT CONDITION FOR DIFFERENTIABILITY**

If $f$ is a function of $x$ and $y$, where $f_x$ and $f_y$ are continuous in an open region *R,* then $f$ is differentiable on *R.*

**THEOREM: DIFFERENTIABILITY IMPLIES CONTINUITY**

If a function of $x$ and $y$ is differentiable at $(x_0, y_0)$ then it is continuous at $(x_0, y_0)$.

Example 3: A triangle is measured and two adjacent sides are found to be 3 inches and 4 inches long, with an included angle of $\frac{\pi}{4}$. The possible errors in measurement are $\frac{1}{16}$ inch for the sides and 0.02 radian for the angle. Approximate the maximum possible error in the computation of the area.

Example 4: Show that the function $f(x,y)=x^2+y^2$ is continuous by finding values for $\varepsilon_1$ and $\varepsilon_2$ as designated in the definition of differentiability, and verify that both $\varepsilon_1$ and $\varepsilon_2 \to 0$ as $(\Delta x,\Delta y)\to(0,0)$.

# Section 13.5 Chain Rules For Functions of Several Variables

When you are done with your homework you should be able to…

- π Use the chain rules for functions of several variables
- π Find partial derivatives implicitly

Warm-up: A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is eight feet deep.

**THEOREM: CHAIN RULE: ONE INDEPENDENT VARIABLE**

Let $w = f(x, y)$, where $f$ is a differentiable function $x$ and $y$. If $x = g(t)$ and $y = h(t)$, where $g$ and $h$ are differentiable functions of $t$, then $w$ is a differentiable function of $t$, and

This can be extended to any number of variables. If $w = f(x_1, x_2, \ldots, x_n)$, you would have

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial w}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial w}{\partial x_n}\frac{dx_n}{dt}$$

Example 1: Find $\frac{\partial w}{\partial t}$ (1) using the appropriate chain rule and (2) by converting *w* to a function of *t* before differentiating.

a. $w=\cos(x-y),\ \ x=t^2,\ \ y=1$

b. $w=xyz,\ \ x=t^2,\ \ y=2t,\ \ z=e^{-t}$

**THEOREM: CHAIN RULE: TWO INDEPENDENT VARIABLES**

Let $w = f(x, y)$, where $f$ is a differentiable function $x$ and $y$. If $x = g(s,t)$ and $y = h(s,t)$, such that the first partials $\partial x/\partial s$, $\partial x/\partial t$, $\partial y/\partial s$, and $\partial y/\partial t$ all exist, then $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ exist and are given by

This can be extended to any number of variables. If $w$ is a differentiable function of the $n$ variables where each $x_1, x_2, \ldots, x_n$ is a differentiable function of the $m$ variables $t_1, t_2, \ldots, t_m$, then for $w = f(x_1, x_2, \ldots, x_n)$, you would have

$$\frac{\partial w}{\partial t_1} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_1}$$
$$\frac{dw}{dt_2} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_2}$$
$$\vdots$$
$$\frac{\partial w}{\partial t_m} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m}$$

Example 2: Find $\partial w/\partial s$ and $\partial w/\partial t$ using the appropriate chain rule, and evaluate each partial derivative at the given values of *s* and *t.*

| Function | Point |
|---|---|
| $w = y^3 - 3x^2 y$ <br> $x = e^s, \quad y = e^t$ | $s = 0, \quad t = 1$ |

**THEOREM: CHAIN RULE: IMPLICIT DIFFERENTIATION**

If the equation $F(x,y)=0$ defines $y$ implicitly as a differentiable function of $x$, then

$$\frac{dy}{dx}=-\frac{F_x(x,y)}{F_y(x,y)},\quad F_y(x,y)\neq 0.$$

If the equation $F(x,y,z)=0$ defines $z$ implicitly as a differentiable function of $x$ and $y$, then

$$\frac{\partial z}{\partial x}=-\frac{F_x(x,y,z)}{F_z(x,y,z)} \text{ and } \frac{\partial z}{\partial y}=-\frac{F_y(x,y,z)}{F_z(x,y,z)},\quad F_z(x,y,z)\neq 0.$$

Example 3: Differentiate implicitly to find $\frac{dy}{dx}$.

$\cos x+\tan xy+5=0$

Example 4: Differentiate implicitly to find the first partial derivatives of $z$.

$x\ln y+y^2z+z^2=8$

Example 5: The radius of a right circular cone is increasing at a rate of 6 inches per minute, and the height is decreasing at a rate of 4 inches per minute. What are the rates of change of the volume and surface area when the radius is 12 inches and the height is 36 inches?

# Section 13.6 Directional Derivatives and Gradients

When you are done with your homework you should be able to…

- π Find and use directional derivatives of a function of two variables
- π Find the gradient of a function of two variables
- π Use the gradient of a function of two variables in applications
- π Find directional derivatives and gradients of functions of three variables

Warm-up: Normalize the following vector (aka find the unit vector):

$\mathbf{v} = 6\mathbf{i} - \mathbf{j}$

Recall that the slope of a surface in the *x*-direction is given by _______________ and the slope of a surface in the *y*-direction is given by _________________. In this section, we will find that these two _________________ _____________ can be used to find the slope in any direction.

**DEFINITION: DIRECTIONAL DERIVATIVE**

Let $f$ be a function of two variables $x$ and $y$ and let $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ be a unit vector. Then the **directional derivative of *f* in the direction of u**, denoted by $D_{\mathbf{u}}f$, is

provided the limit exists.

**THEOREM: DIRECTIONAL DERIVATIVE**

If $f$ is a differentiable function of $x$ and $y$, then the directional derivative of $f$ in the direction of the unit vector $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ is

There are infinitely many directional derivatives to a surface at a given point—one for each direction specified by $\mathbf{u}$.

Example 1: Find the directional derivative of the following functions at the given point and direction.

a. $f(x,y) = x^3 - y^3$, at the point $P(4,3)$, in the direction $\mathbf{v} = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$

b. $f(x,y)=\cos(x+y)$, at the point $P(0,\pi)$, in the direction $Q\left(\frac{\pi}{2},0\right)$

**DEFINITION: GRADIENT OF A FUNCTION OF TWO VARIABLES**

Let $z=f(x,y)$, be a function of $x$ and $y$ such that $f_x$ and $f_y$ exist. Then the **gradient of *f***, denoted by $\nabla f(x,y)$, is the vector

$\nabla f$ is read as "del *f*". Another notation for the gradient is $\mathbf{grad}\, f(x,y)$.

Example 2: Find the gradient of $f(x,y)=\ln(x^2-y)$, at the point $(2,3)$.

**THEOREM: ALTERNATIVE FORM OF THE DIRECTIONAL DERIVATIVE**

If $f$ is a differentiable function of $x$ and $y$, then the directional derivative of $f$ in the direction of the unit vector $\mathbf{u}$ is

Example 3: Use the gradient to find the directional derivative of the function $f(x,y)=\sin 2x\cos y$ at the point $P(0,0)$ in the direction of $Q\left(\frac{\pi}{2},\pi\right)$.

**THEOREM: PROPERTIES OF THE GRADIENT**

Let $f$ be differentiable at the point $(x, y)$.

If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y) = 0$ for all $\mathbf{u}$

The direction of *maximum* increase of $f$ is given by ___________________. The maximum value of $D_{\mathbf{u}} f(x, y)$ is ________________________________.

The direction of *minimum* increase of $f$ is given by _____________________. The minimum value of $D_{\mathbf{u}} f(x, y)$ is ________________________________.

Example 4: The surface of a mountain is modeled by the equation $h(x, y) = 5000 - 0.001x^2 - 0.004y^2$. A mountain climber is at the point $(500, 300, 4390)$. In what direction should the climber move in order to ascend at the greatest rate?

**THEOREM: GRADIENT IS NORMAL TO LEVEL CURVES**

If $f$ is differentiable at $(x_0, y_0)$ and $\nabla f(x, y) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is __________________ to the _____________ curve through ________________.

**THEOREM: PROPERTIES OF THE GRADIENT**

Let $f$ be a function of $x, y,$ and $z$, with continuous first partial derivatives. The **directional derivative of *f*** in the direction of a unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is given by

The **gradient of *f*** is defined to be

1. ________________________________________

2. If $\nabla f(x, y, z) = \mathbf{0},$ then ____________________ for all _______

3. The direction of _______________ increase of $f$ is given by _______________. The maximum ________ of $D_{\mathbf{u}} f(x, y, z)$ is ____________________.

4. The direction of *minimum* ______________ of $f$ is given by ______________. The minimum value of $D_{\mathbf{u}} f(x, y, z)$ is _____________________.

Example 5: Find the gradient of the function $w = xy^2z^2$ and the maximum value of the directional derivative at the point $(2,1,1)$.

# Section 13.7 Tangent Planes and Normal Lines

When you are done with your homework you should be able to…

- π Find equations of tangent planes and normal lines to surfaces
- π Find the angle of inclination of a plane in space
- π Compare the gradients $\nabla f(x,y)$ and $\nabla F(x,y)$

Warm-up: Find the general equation of the plane containing the points *P*(2, 1, 1), *Q*(0, 4, 1), and *R*(-2, 1, 4).

**DEFINITION OF TANGENT PLANE AND NORMAL LINE**

Let $F$ be differentiable at the point $P(x_0, y_0, z_0)$ on the surface given by $F(x,y,z)=0$ such that $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$.

1. The plane through $P$ that is normal to $\nabla F(x_0, y_0, z_0)$ is called the ______________ plane to___ at ___.
2. The line through $P$ having the direction of $\nabla F(x_0, y_0, z_0)$ is called the normal ________ to ___ at ___.

Note: We've been using ________________ for a surface _______. Rewrite as ______________=__________________.

_____ is the _____________ __________________ of ____ given by ___________________________.

Example 1: Find a unit normal vector to the surface at the given point. *HINT:* normalize the gradient vector $\nabla F(x, y, z)$.

$x^2 + y^2 + z^2 = 11$, at the point $P(3,1,1)$

**THEOREM: EQUATION OF TANGENT PLANE**

If $F$ is differentiable at $(x_0, y_0, z_0)$, then an equation of the tangent plane to the surface is given by $F(x, y, z) = 0$ at $(x_0, y_0, z_0)$ is

Example 2: Find an equation of the tangent plane to the surface at the given point.

$h(x,y) = \ln\sqrt{x^2+y^2}$, at the point $P(3,4,\ln 5)$

Example 3: Find an equation of the tangent plane and find symmetric equations of the normal line to the surface at the given point.

$z = \arctan\frac{y}{x}$, at the point $\left(1,1,\frac{\pi}{4}\right)$

Example 4: Find the path of a heat-seeking particle placed at the point in space $(2,2,5)$ with a temperature field $T(x,y,z)=100-3x-y-z^2$.

**THE ANGLE INCLINATION OF A PLANE**

$$\cos\theta = \frac{|\mathbf{n}\cdot\mathbf{k}|}{\|\mathbf{n}\|}$$

**THEOREM: GRADIENT IS NORMAL TO LEVEL SURFACES**

If $F$ is differentiable at $(x_0,y_0,z_0)$ and $\nabla F(x_0,y_0,z_0)\neq \mathbf{0}$, then $\nabla F(x_0,y_0,z_0)$ is ______________ to the ____________ through $(x_0,y_0,z_0)$.

# Section 13.8 Extrema of Functions of Two Variables

When you are done with your homework you should be able to…

- π Find the absolute and relative extrema of a function of two variables
- π Use the Second Partials Test to find relative extrema of a function of two variables

Warm-up: Consider the function $f(x) = \sin x \cos x$ on the interval $(0, \pi)$.

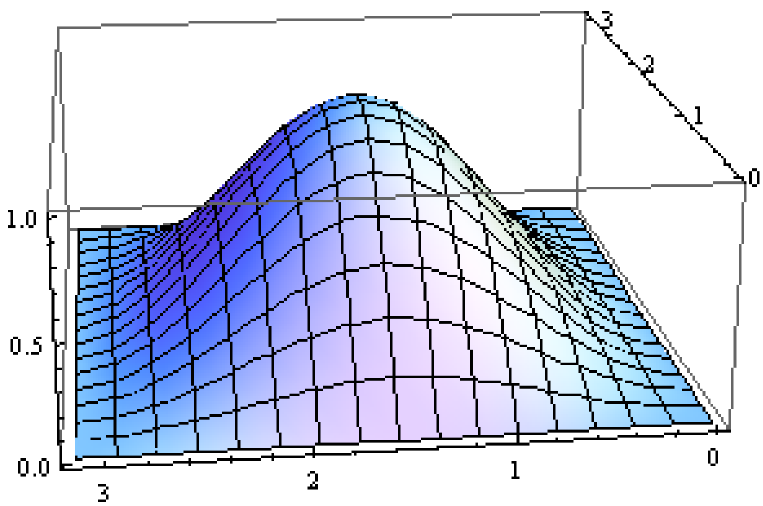
1. Find the critical numbers.

2. Apply the theorem which tests for increasing and decreasing intervals.

3. Find the open interval(s) on which the function is
   a. Increasing

   b. Decreasing

4. Apply the First Derivative test to identify all relative extrema. Give your result(s) as an ordered pair.




Plot3D[Sin[x]Sin[y]^2, {x, 0, Pi}, {y, 0, Pi}]

**THEOREM: EXTREME VALUE THEOREM**

Let $f$ be a continuous function of two variables $x$ and $y$ defined on a closed bounded region $R$ in the *xy*-plane.

1. There is at least one point in $R$ where $f$ takes on a minimum value.
2. There is at least one point in $R$ where $f$ takes on a maximum value.

**DEFINITION: RELATIVE EXTREMA**

Let $f$ be a function defined on a region $R$ containing $(x_0, y_0)$.

1. The function $f$ has a **relative minimum** at $(x_0, y_0)$ if ________________________________for all $x$ and $y$ in an *open* disk containing $(x_0, y_0)$.
2. The function $f$ has a **relative maximum** at $(x_0, y_0)$ if ________________________________for all $x$ and $y$ in an *open* disk containing $(x_0, y_0)$.

**DEFINITION: CRITICAL POINT**

Let $f$ be defined on an open region $R$ containing $(x_0, y_0)$. The point $(x_0, y_0)$ is a **critical point** of $f$ if one of the following is true.

1. ______________________ and __________________________
2. ______________________ or _______________________ does not exist.

**THEOREM: RELATIVE EXTREMA OCCUR ONLY AT CRITICAL POINTS**

If $f$ has a relative ________________________ at $(x_0, y_0)$ on an open region $R$, then $(x_0, y_0)$ is a critical point of $f$.

**THEOREM: SECOND PARTIALS TEST**

Let $f$ have continuous partial derivatives on an open region containing a point $(a,b)$ for which $f_x(a,b) = 0$ and $f_y(a,b) = 0$. To test for relative extrema of $f$, consider the quantity

$d = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$.

1. If ___________ and _____________________, then $f$ has a **relative minimum** at $(a,b)$.
2. If ___________ and _____________________, then $f$ has a **relative maximum** at $(a,b)$.
3. If ___________, then _______________________ is a **saddle point**.
4. The test is inconclusive if ____________.

Example 1: Examine the function for relative extrema and saddle points.

$g(x, y) = xy$

Example 2: Find the critical points and test for relative extrema. List the critical points for which the Second Partials Test fails.

$$f(x,y) = x^3 + y^3 - 6x^2 + 9y^2 + 12x + 27y + 19$$

Example 3: A function $f$ has continuous second partial derivatives on an open region containing the critical point $(a,b)$. If $f_{xx}(a,b)$ and $f_{yy}(a,b)$ have opposite signs, what is implied?

# Section 13.9 Applications of Extrema

When you are done with your homework you should be able to…

- $\pi$ Solve optimization problems involving functions of several variables

Warm-up: Examine the function $g(x,y)=120x+120y-xy-x^2-y^2$ for relative extrema and saddle points.

Example 1: Find the minimum distance from the point $(1,2,3)$ to the plane $2x+3y+z=12$. (HINT: To simplify the computations, minimize the square of the distance).

Example 2: Find three positive numbers *x, y,* and *z* which have a sum of 1 and the sum of the squares is a minimum.

Example 3: The material for constructing the base of an open box costs 1.5 times as much per unit area as the material for constructing the sides. For a fixed amount of money *C*, find the dimensions of the box of largest volume that can be made.

Example 4: A retail outlet sells two types of riding lawn mowers, the prices of which are $p_1$ and $p_2$. Find $p_1$ and $p_2$, so as to maximize total revenue, where $R = 515p_1 + 805p_2 + 1.5p_1p_2 - 1.5{p_1}^2 - {p_2}^2$.

# Section 13.10 Lagrange Multipliers

When you are done with your homework you should be able to…

- π Understand the method of Lagrange multipliers
- π Use Lagrange multipliers to solve constrained optimization problems

Many optimization problems have ____________________, or _____________________, on the values that can be used to produce the ___________________ solution.

**THEOREM: LAGRANGE'S THEOREM**

Let $f$ and $g$ have continuous first partial derivatives such that $f$ has an extremum at a point $(x_0, y_0)$ on the smooth constraint curve $g(x,y)=c$. If $\nabla g(x_0,y_0) \neq \mathbf{0}$, then there is a real number _____ such that

**METHOD OF LAGRANGE MULTIPLIERS**

Let $f$ and $g$ satisfy the hypothesis of Lagrange's Theorem, and let $f$ have a minimum or maximum subject to the constraint $g(x,y)=c$. To find the minimum or maximum of $f$, use these steps. containing $(x_0, y_0)$.

1. Simultaneously, solve the equations ____________________________ and _______________________ by solving the system of equations.

2. Evaluate $f$ at each ________________ point obtained in the first step. The greatest value yields the __________________ of $f$ subject to the constraint _______________________, and the _____________ value yields the ________________________ of $f$ subject to the constraint $g(x,y)=c$.

Example 1: Use Lagrange multipliers to minimize $f(x,y)=2x+y$ with the constraint $x+y=10$, assuming $x$ and $y$ are positive.

Example 2: Use Lagrange multipliers to find the minimum distance from the curve, $y = x^2$ to the point $(-3,0)$.

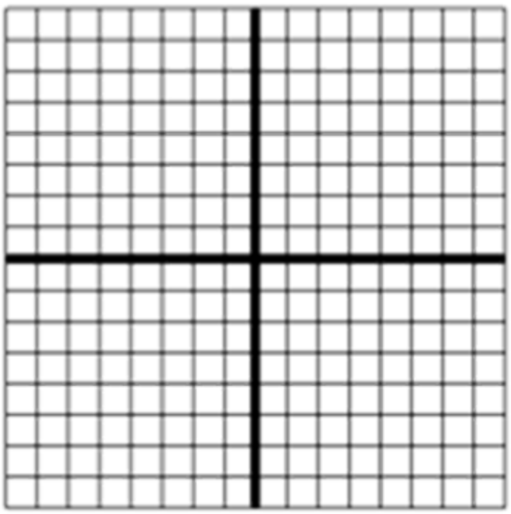
# Chapter 14 Multiple Integration

## Section 14.1 Iterated Integrals and Area In the Plane

When you are done with your homework you should be able to…

- π Evaluate an iterated integral
- π Use an iterated integral to find the area of a plane region

Warm-up: Sketch the region bounded by the graphs $x = \cos y,\ x = \frac{1}{2},\ \frac{\pi}{3} \le y \le \frac{7\pi}{3}$. Then find the area.



**INTEGRALS OF FUNCTIONS OF TWO VARIABLES**

When integrating a function of two variables with respect to *x*, you hold *y* constant:

$$\int_{h_1(y)}^{h_2(y)} f_x(x,y)\,dx = f(x,y)\Big]_{h_1(y)}^{h_2(y)} = f(h_2(y),y) - f(h_1(y),y).$$

When integrating a function of two variables with respect to *y*, you hold *x* constant:

$$\int_{g_1(x)}^{g_2(x)} f_y(x,y)\,dy = f(x,y)\Big]_{g_1(x)}^{g_2(x)} = f(x,g_2(x)) - f(x,g_1(x)).$$

Example 1: Evaluate the following integrals.

a. $\displaystyle\int_{x}^{x^2} \frac{y}{x}\,dy$

b. $\displaystyle\int_{y}^{\pi/2} \sin^3 x \cos y\,dx$

**ITERATED INTEGRALS**

When evaluating the integral of an integral, it is called an **iterated integral**.

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b f(x,y)\Big]_{g_1(x)}^{g_2(x)} dx$$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d f(x,y)\Big]_{h_1(y)}^{h_2(y)} dy$$

Example 2: Evaluate the following iterated integrals.

a. $\int_0^1 \int_0^2 (x+y)\,dy\,dx$

b. $\int_1^4 \int_1^{\sqrt{x}} 2ye^{-x}\,dy\,dx$

c. $\int_0^3 \int_0^\infty \frac{x^2}{1+y^2}\, dy\, dx$

**AREA OF A REGION IN THE PLANE**

If *R* is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where $g_1$ and $g_2$ are continuous on $[a,b]$, then the area of *R* is given by

(vertically simple)

(horizontally simple)

Example 3: Use an iterated integral to find the area of the region bounded by the graphs of $y = x,\ y = 2x,\ x = 2.$

Example 4: Sketch the region *R* whose area is given by the iterated integral. Then switch the order of integration and show that both orders yield the same area. $\int_{-2}^{2}\int_{0}^{4-y^2} dx\,dy$

# Section 14.2 Double Integrals and Volume

When you are done with your homework you should be able to…

- π Use a double integral to represent the volume of a solid region
- π Use properties of double integrals
- π Evaluate a double integral as an iterated integral

Warm-up: Evaluate the iterated integral $\int_0^{\pi}\int_0^{\pi/2} \sin^2 x \cos^2 y \, dy \, dx$.

ACTIVITY: The table below shows values of a function $f$ over a square region $R$. Divide the region into 16 equal squares and select $(x_i, y_i)$ to be the point in the *i*th square closest to the origin. Compare this approximation with that obtained by using the point in the *i*th square furthest from the origin.

| $x$ \ $y$ | **0** | **1** | **2** | **3** | **4** |
|---|---|---|---|---|---|
| **0** | 32 | 31 | 28 | 23 | 16 |
| **1** | 31 | 30 | 27 | 22 | 15 |
| **2** | 28 | 27 | 24 | 19 | 12 |
| **3** | 23 | 22 | 19 | 14 | 7 |
| **4** | 16 | 15 | 12 | 7 | 0 |

**DEFINITION: DOUBLE INTEGRAL**

If $f$ is defined on a closed, bounded region $R$ in the *xy*-plane, then the **double integral of *f* over *R*** is given by

provided the limit exists. If the limit exists, then $f$ is **integrable** over $R$.

**VOLUME OF A SOLID REGION**

If $f$ is integrable over a plane region $R$ and $f(x,y) \geq 0$ for all $(x,y)$ in $R$, then the volume of the solid region that lies above $R$ and below the graph of $f$ is defined as

Example 1: Sketch the region $R$ and evaluate the iterated integral $\int_R \int f(x,y)\,dA$.

$$\int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} x^2 y^2 \,dx\,dy$$

**PROPERTIES OF DOUBLE INTEGRALS**

Let $f$ and $g$ be continuous over a closed, bounded plane region $R$, and let $c$ be a constant.

1. $\int_R \int cf(x,y)\,dA = c\int_R \int f(x,y)\,dA$
2. $\int_R \int [f(x,y) \pm g(x,y)]\,dA = \int_R \int f(x,y)\,dA \pm \int_R \int g(x,y)\,dA$
3. $\int_R \int f(x,y)\,dA \geq 0, \quad \text{if } f(x,y) \geq 0$
4. $\int_R \int f(x,y)\,dA \geq \int_R \int g(x,y)\,dA, \quad \text{if } f(x,y) \geq g(x,y)$
5. $\int_R \int f(x,y)\,dA = \int_{R_1} \int f(x,y)\,dA + \int_{R_2} \int f(x,y)\,dA$, where ____ is the _______________ of two

   nonoverlapping subregions ______ and ______.

**THEOREM: FUBINI'S THEOREM**

Let $f$ be continuous on a plane region $R$.

1. If $R$ is defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where $g_1$ and $g_2$ are continuous on $[a,b]$, then

2. If $R$ is defined by $c \leq y \leq d$ and $\mathrm{h}_1(y) \leq x \leq h_2(y)$, where $h_1$ and $h_2$ are continuous on $[c,d]$, then

Example 2: Set up an integrated integral for both orders of integration, and use the more convenient order to evaluate over the region $R$.

$$\int_R \int xe^y \, dA,$$

$R$: triangle bounded by $y = 4 - x,\ y = 0,\ x = 0$

Example 3: Set up a double integral to find the volume of the solid bounded by the graphs of the equations $x^2 + z^2 = 1,\ y^2 + z^2 = 1,$ first octant.

Example 4: Find the average value of $f(x, y)$ over the region *R* where

$$\text{Average value} = \frac{1}{A}\int_R\int f(x,y)\,dA, \text{ where } A \text{ is the area of } R.$$

$f(x,y) = xy.$

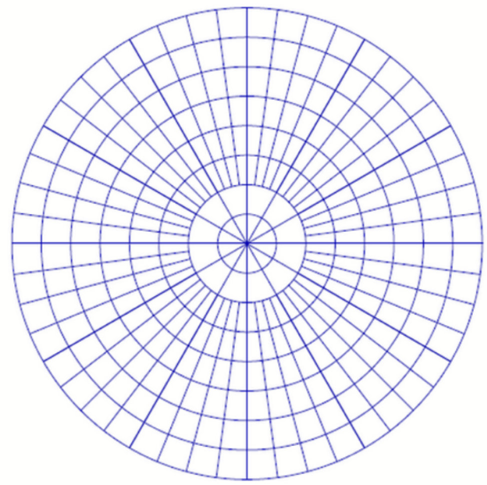
$R$: rectangle with vertices $(0,0)$, $(4,0)$, $(4,2)$ and $(0,2)$.

# Section 14.3 Change of Variables: Polar Coordinates

When you are done with your homework you should be able to…

- $\pi$ Write and evaluate double integrals in polar coordinates

Warm-up: Find the area of the region inside $r = 3\sin\theta$ and outside $r = 2 - \sin\theta$..



Recall:

$x = r\cos\theta$ and $y = r\sin\theta$

$r^2 = x^2 + y^2$ and $\tan\theta = \dfrac{y}{x}$

**THEOREM: CHANGE OF VARIABLES IN POLAR FORM**

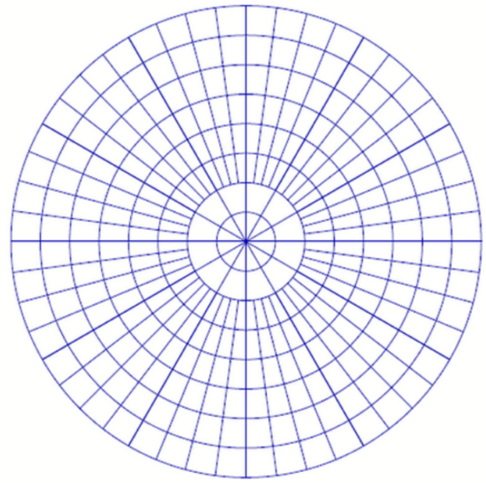
Let $R$ be a plane region consisting of all points $(x, y) = (r\cos\theta, r\sin\theta)$ satisfying the conditions $0 \le g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, where $0 \le (\beta - \alpha) \le 2\pi$. If $g_1$ and $g_2$ are continuous on $[\alpha, \beta]$ and $f$ is continuous on $R$, then

Example 1: Evaluate the double integral $\int_R \int f(r,\theta)\, dA$ and sketch the region $R$.

$$\int_0^{\pi/4} \int_0^4 r^2 \sin\theta \cos\theta \, dr\, d\theta$$

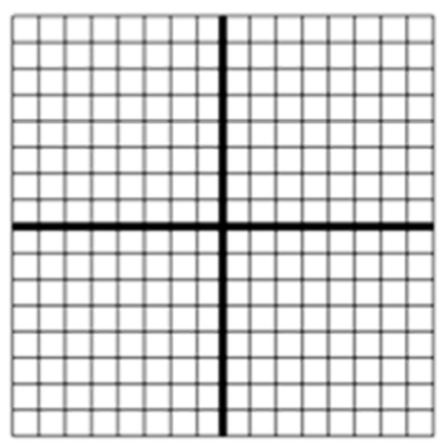
Example 2: Evaluate the iterated integral by converting to polar coordinates.

$$\int_0^2 \int_y^{\sqrt{8-y^2}} \sqrt{x^2+y^2}\, dx\, dy$$



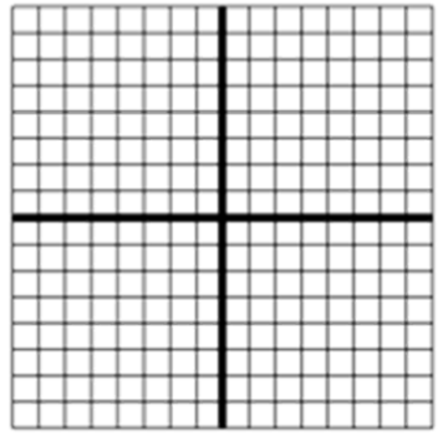
Example 3: Use polar coordinates to set up and evaluate the double integral $\int_R \int f(x,y)\, dA$.

$$f(x,y) = e^{-\left(x^2+y^2\right)/2},\ R: x^2+y^2 \le 25,\ x^2 \ge 0\ .$$



Example 4: Use a double integral in polar coordinates to find the volume of the solid bounded by the graphs of the equations

$z = \ln\left(x^2 + y^2\right),\ z = 0,\ x^2 + y^2 \geq 1,\ x^2 + y^2 \leq 4$



# Section 14.5 Surface Area

When you are done with your homework you should be able to…

- π Use a double integral to find the area of a surface

Warm-up: Find the area of the parallelogram with vertices $A=(2,-3,1)$, $B=(6,5,-1)$, $C=(3,-6,4)$ and $D=(7,2,2)$.

Hint: Section 11.4

**DEFINITION: SURFACE AREA**

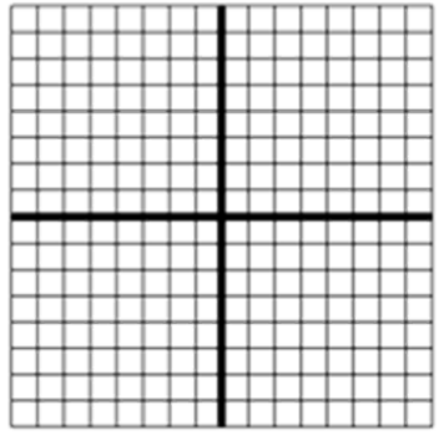
If $f$ and its first partial derivatives are continuous on the closed region $R$ in the *xy*-plane, then the **<u>area of the surface *S*</u>** given by $z=f(x,y)$ over $R$ is given by

$$\text{Surface Area} = \int_R\int dS$$
$$= \int_R\int \sqrt{1+\left[f_x(x,y)\right]^2+\left[f_y(x,y)\right]^2}\,dA$$

Example 1: Find the area of the surface given by $z = f\left(x, y\right)$ over the region $R$.

$f\left(x, y\right) = 15 + 2x - 3y$

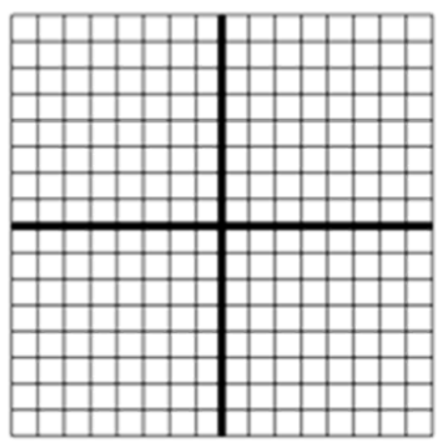
$R$: square with vertices $\left(0,0\right)$, $\left(3,0\right)$, $\left(0,3\right)$, $\left(3,3\right)$



Example 2: Find the area of the surface given by $z = f\left(x, y\right)$ over the region $R$.

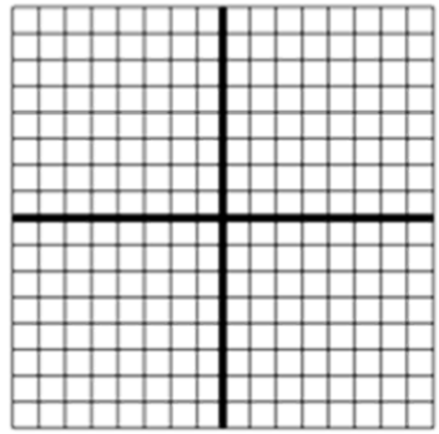
$f\left(x, y\right) = xy$

$R = \left\{\left(x, y\right) \mid x^2 + y^2 \le 16\right\}$



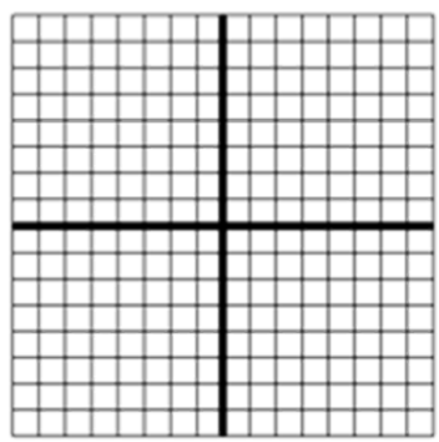
Example 3: Find the area of the surface.

The portion of the cone $z = 2\sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 = 4$ .



Example 4: Set up a double integral that gives the area of the surface on the graph of

$f(x,y) = e^{-x}\sin y,\ R = \{(x,y) \mid 0 \le x \le 4,\ 0 \le y \le x\}.$



# Section 14.6 Triple Integrals and Applications

When you are done with your homework you should be able to…

- π Use a triple integral to find the volume of a solid region
- π Find the center of mass and moments of inertia of a solid region

Warm-up: Set up a double integral to find the volume of the solid bounded by the graphs of the equations $z=\dfrac{1}{1+y^2}$, $x=0$, $x=2$ and $y\geq 0$.

**DEFINITION: TRIPLE INTEGRAL**

If $f$ is continuous over a bounded solid region $Q$, then the **triple integral of *f* over *Q*** is defined as

Provided the limit exists. The **volume** of the solid region $Q$ is given by

**THEOREM: EVALUATION BY ITERATED INTEGRALS**

Let $f$ be continuous on a solid region $Q$ defined by $a \le x \le b,\ h_1(x) \le y \le h_2(x),\ g_1(x,y) \le z \le g_2(x,y)$

where $h_1,\ h_2,\ g_1,$ and $g_2$ are continuous functions. Then

Example 1: Evaluate the iterated integral.

$$\int_1^4 \int_1^{e^2} \int_0^{1/(xz)} \ln z \, dy \, dz \, dx$$

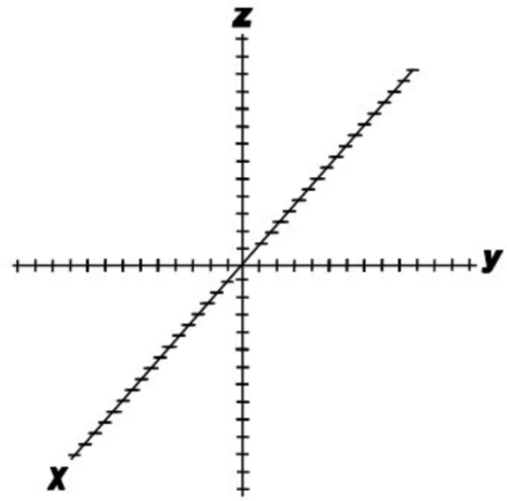
Example 2: Set up a triple integral for the volume of the solid.

The solid that is the common interior below the sphere $x^2 + y^2 + z^2 = 80$ and above the paraboloid $z = \frac{1}{2}\left(x^2 + y^2\right)$

Example 3: Sketch the solid whose volume is given by the iterated integral and rewrite the integral using the indicated order of integration.

$$\int_0^2 \int_{2x}^4 \int_0^{\sqrt{y^2-4x^2}} dz\,dy\,dx$$

Rewrite using the order $dxdydz$




Example 4: List the six possible orders of integration for the triple integral over the solid region $Q$ $\iiint\limits_Q xyz\,dV$.

$$Q = \{(x, y, z) : 0 \le x \le 2,\ x^2 \le y \le 4,\ 0 \le z \le 6\}$$

# Section 14.7 Triple Integrals In Other Coordinates

When you are done with your homework you should be able to…

- π Write and evaluate a triple integral in cylindrical coordinates
- π Write and evaluate a triple integral in spherical coordinates

Warm-up:

1. Find an equation in cylindrical coordinates for the equation $x^2 + y^2 - 3z^2 = 0$.

2. Find an equation in spherical coordinates for the equation $x^2 + y^2 - 3z^2 = 0$.

**RECALL: CYLINDRICAL COORDINATES**

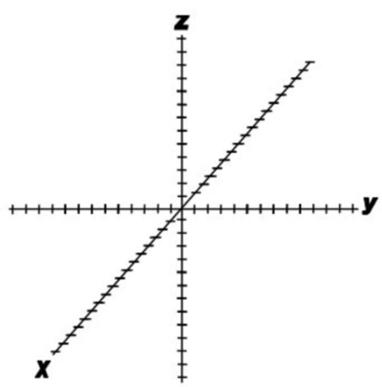
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

**ITERATED FORM OF THE TRIPLE INTEGRAL IN CYLINDRICAL FORM:**

If $Q$ is a solid region whose projection $R$ onto the *xy*-plane can be described in polar coordinates, that is, $Q = \{(x,y,z) : (x,y) \text{ is in } R,\ h_1(x,y) \le z \le h_2(x,y)\}$ and $R = \{(r,\theta) : \theta_1 \le \theta \le \theta_2,\ g_1(\theta) \le r \le g_2(\theta)\}$, and if $f$ is a continuous function on the solid $Q$, you can write the triple integral of $f$ over $Q$ as

**RECALL: SPHERICAL COORDINATES**

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$



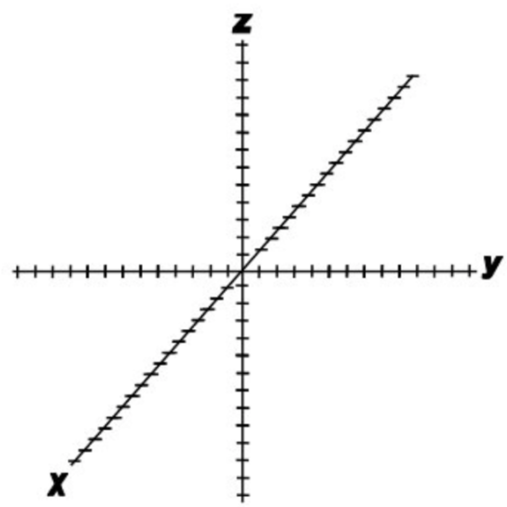

**ITERATED FORM OF THE TRIPLE INTEGRAL IN SPHERICAL FORM:**

Example 1: Evaluate the iterated integral.

$$\int_0^{\pi/2}\int_0^{\pi}\int_0^{2} e^{-\rho^3}\rho^2 \, d\rho \, d\theta \, d\phi$$

Example 2: Sketch the solid region whose volume is given by the iterated integral and evaluate the iterated integral.

$$\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{3-r^2} r \, dz \, dr \, d\theta$$

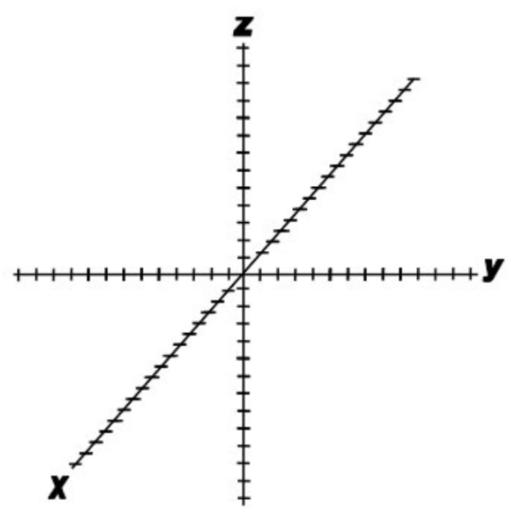


Example 3: Convert the integral from rectangular coordinates to both cylindrical and spherical coordinates, and evaluate the simplest iterated integral.

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{16-x^2-y^2}} \sqrt{x^2+y^2}\, dz\, dy\, dx$$

Example 4: Use cylindrical coordinates to find the volume of the solid.

Solid inside $x^2+y^2+z^2=16$ and outside $z=\sqrt{x^2+y^2}$



Example 5: Convert the integral from rectangular coordinates to both cylindrical and spherical coordinates, and evaluate the simplest iterated integral.

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$$

# Section 14.8 Change of Variables: Jacobians

When you are done with your homework you should be able to…

- π Understand the concept of a Jacobian
- π Use a Jacobian to change variables in a double integral

For the single integral ______________________ you can change variables by letting _______________, so that

____________________, and obtain ________________________________________________where _____________ and

_____________. Note that the change of variables introduces an additional factor _______ into the integrand. This also

occurs in the case of double integrals

where the change of variables ____________________ and ____________________ introduces a factor called the

______________________ of ____ and ____ with respect to ____ and ____.

**DEFINITION OF THE JACOBIAN**

If $x = g(u,v)$ and $y = h(u,v)$, then the Jacobian of $x$ and $y$ with respect to $u$ and $v$, denoted by $\partial(x,y)/\partial(u,v)$, is

Example 1: Find the Jacobian $\partial(x,y)/\partial(u,v)$ for the indicated change of variables.

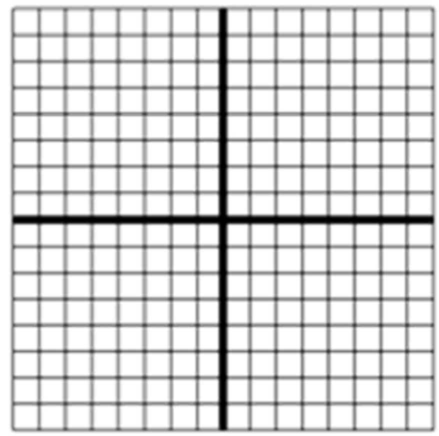
$x = uv - 2u,\ y = uv$

**THEOREM: CHANGE OF VARIABLES FOR DOUBLE INTEGRALS**

Let $R$ be a vertically or horizontally simple region in the *xy*-plane, and let $S$ be a vertically or horizontally simple region in the *uv*-plane. Let $T$ from $S$ to $R$ be given by $T(u,v)=(x,y)=(g(u,v),h(u,v))$ where $g$ and $h$ have continuous partial derivatives. Assume that $T$ is one-to-one except possibly on the boundary of $S$. If $f$ is continuous on $R$, and $\partial(x,y)/\partial(u,v)$ is nonzero on $S$, then

Example 2: Use the indicated change of variables to evaluate the double integral.

$$\int_R\int 4(x+y)e^{x-y}\,dA$$



# Chapter 15 Vector Analysis

## Section 15.1 Vector Fields

When you are done with your homework you should be able to…

- $\pi$ Understand the concept of a vector field
- $\pi$ Determine whether a vector field is conservative
- $\pi$ Find the curl of a vector field
- $\pi$ Find the divergence of a vector field

Warm-up: A 48,000-pound truck is parked on a $10^\circ$ slope. Assume the only force to overcome is that due to gravity.

a. Find the force required to keep the truck from rolling down the hill.

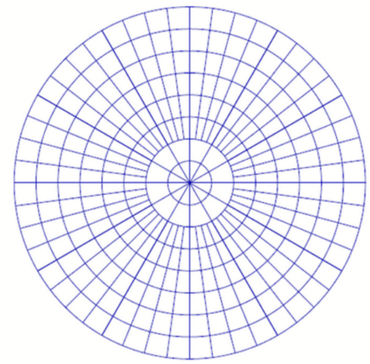
b. Find the force perpendicular to the hill.

**DEFINITION OF VECTOR FIELD**

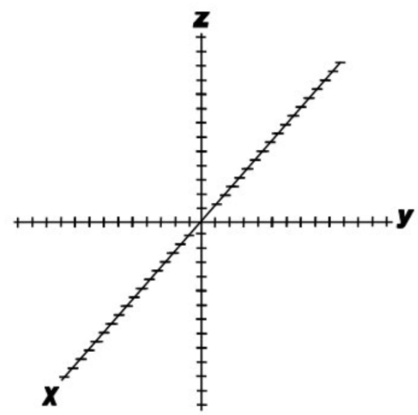
Let $M$ and $N$ be functions of two variables $x$ and $y$, defined on a plane region $R$. The function $\mathbf{F}$ defined by $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$ is called a **vector field over *R***. (plane)

Let $M$, $N$, and $P$ be functions of three variables $x$, $y$ and $z$, defined on a solid region $Q$. The function $\mathbf{F}$ defined by $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is called a **vector field over *Q***. (space)

Example 1: Sketch several representative vectors in the vector field $\mathbf{F}(x,y) = x\mathbf{i} - y\mathbf{j}$.



Example 2: Sketch several representative vectors in the vector field $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.




**DEFINITION OF INVERSE SQUARE FIELD**

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be a position vector. The vector field $\mathbf{F}$ is an **inverse square field** if

where *k* is a real number and $\mathbf{u} = \dfrac{\mathbf{r}}{\|\mathbf{r}\|}$ is a unit vector in the direction of $\mathbf{r}$.

**DEFINITION OF CONSERVATIVE VECTOR FIELD**

A vector field $\mathbf{F}$ is called **conservative** if there exists a differentiable function $f$ such that $\mathbf{F} = \nabla f$. The function $f$ is called the **potential function** for $\mathbf{F}$.

Example 3: Find the gradient vector field for the scalar function. That is, find the conservative vector field for the potential function.

$$f\left(x,y,z\right)=\frac{y}{z}+\frac{z}{x}-\frac{xz}{y}$$

**THEOREM: TEST FOR CONSERVATIVE VECTOR FIELD IN THE PLANE**

Let $M$ and $N$ have continuous first partial derivatives on an open disk $R$. The vector field given by $\mathbf{F}\left(x,y\right)=M\mathbf{i}+N\mathbf{j}$ is conservative if and only if $\dfrac{dN}{dx}=\dfrac{dM}{dy}$.

Example 4: Determine whether the vector field is conservative. If it is, find a potential function for the vector field.

$$\mathbf{F}\left(x,y\right)=\frac{1}{y^2}\left(y\mathbf{i}-2x\mathbf{j}\right)$$

**DEFINITION OF A CURL OF A VECTOR FIELD**

The curl of $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is

$$\begin{aligned}\textbf{curl }\mathbf{F}(x,y,z) &= \nabla \times \mathbf{F}(x,y,z)\\ &= \left(\frac{dP}{dy} - \frac{dN}{dz}\right)\mathbf{i} - \left(\frac{dP}{dx} - \frac{dM}{dz}\right)\mathbf{j} + \left(\frac{dN}{dx} - \frac{dM}{dy}\right)\mathbf{k}\end{aligned}$$

Example 5: Find curl $\mathbf{F}$ for the vector field $\mathbf{F}(x,y,z) = e^{-xyz}(\mathbf{i} + \mathbf{j} + \mathbf{k})$ at the point $(3,2,0)$.

**THEOREM: TEST FOR CONSERVATIVE VECTOR FIELD IN SPACE**

Suppose that $M$, $N$ and $P$ have continuous first partial derivatives on an open sphere $Q$ in space. The vector field given by $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative if and only if $\textbf{curl }\mathbf{F}(x,y,z) = \mathbf{0}$.

That is, $\mathbf{F}$ is conservative if and only if

$$\frac{dP}{dy} = \frac{dN}{dz},\ \frac{dP}{dx} = \frac{dM}{dz},\ \text{and}\ \frac{dN}{dx} = \frac{dM}{dy}.$$

Example 6: Determine whether the vector field is conservative. If it is, find a potential function for the vector field.

$$\mathbf{F}(x,y,z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$$

**DEFINITION: DIVERGENCE OF A VECTOR FIELD**

The **divergence** of $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$ is

The **divergence** of $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is

If $\operatorname{div} \mathbf{F}(x,y,z) = 0$, then $\mathbf{F}$ is said to be **divergence free**.

Example 7: Find the divergence of the vector field $\mathbf{F}(x,y,z)=\ln(xyz)(\mathbf{i}+\mathbf{j}+\mathbf{k})$ at the point $(3,2,1)$.

**THEOREM: RELATIONSHIP BETWEEN DIVERGENCE AND CURL**

If $\mathbf{F}(x,y,z)=M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$ is a vector field and $M$, $N$ and $P$ have continuous second partial derivatives, then

- For vector fields representing ___________________ of moving __________________, the ______________________ measures the _______ of ________________ __________ per unit volume.

- In ___________________________________, the study of ______________ ___________________, a velocity field that is ___________________ free is called _______________________________.

- In the study of _______________________ and _______________________________, a vector field that is divergence free is called ____________________________.

# Section 15.2 Line Integrals

When you are done with your homework you should be able to…

- π Understand and use the concept of a piecewise smooth curve
- π Write and evaluate a line integral
- π Write and evaluate a line integral of a vector field
- π Write and evaluate a line integral in differential form

Warm-up:

Represent the plane curve $2x-3y+5=0$ by a vector-valued function.

Determine whether the vector field $\mathbf{F}$ is conservative. If it is, find a potential function for the vector field.

$$\mathbf{F}(x,y,z)=\frac{x}{x^2+y^2}\mathbf{i}+\frac{y}{x^2+y^2}\mathbf{j}+\mathbf{k}$$

**PIECEWISE SMOOTH CURVES:**

The work done by gravity on an object moving between two points in the field is independent of the path taken by the object

One constraint is that the **path** must be a piecewise smooth curve

Recall that a plane curve *C* given by $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j},\quad a\le t\le b$ is smooth if $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous on $[a,b]$ and not simultaneously 0 on $(a,b)$. Similarly, a space curve *C* given by $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k},\quad a\le t\le b$ is smooth if $\frac{dx}{dt}, \frac{dy}{dt}$ and $\frac{dz}{dt}$ are continuous on $[a,b]$ and not simultaneously 0 on $(a,b)$.

A curve *C* is **piecewise smooth** if the interval can be partitioned into a finite number of subintervals, on each of which *C* is smooth.

Example 1: Find a piecewise smooth parametrization of the path *C*.

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

**DEFINITION OF LINE INTEGRAL**

If $f$ is defined in a region containing a smooth curve $C$ of finite length, then the **line integral of *f* along *C*** is given by

$$\int_C f(x,y)\,ds = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(x_i,y_i)\Delta s_i \qquad \text{plane}$$

or

$$\int_C f(x,y,z)\,ds = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(x_i,y_i,z_i)\Delta s_i \qquad \text{space}$$

provided this limit exists.

*To evaluate a line integral over a plane curve *C* given by $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}$, use the fact that

$$ds = \|\mathbf{r}'(t)\|dt = \sqrt{[x'(t)]^2+[y'(t)]^2}\,dt.$$

**THEOREM: EVALUATION OF A LINE INTEGRAL AS A DEFINITE INTEGRAL**

Let $f$ be continuous in a region containing a smooth curve $C$.

If $C$ is given by $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}$, where $a\le t\le b$, then

If $C$ is given by $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$, where $a\le t\le b$, then

Note that if $f(x,y,z)=1$, the line integral gives the arc length of the curve $C$. That is,

$$\int_C 1\,ds = \int_a^b \|r'(t)\|dt = \text{length of curve } C.$$

Example 2: Evaluate the line integral along the given path.

$$\int_C 8xyz\,ds$$
$$C:\ \mathbf{r}(t) = 12t\mathbf{i} + 5t\mathbf{j} + 3\mathbf{k}.$$
$$0 \le t \le 2$$

**DEFINITION OF LINE INTEGRAL OF A VECTOR FIELD**

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve *C* given by $\mathbf{r}(t),\ a \le t \le b$. The **line integral of F on *C*** is given by

Example 3: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $C$ is represented by $\mathbf{r}(t)$

$$\mathbf{F}(x,y,z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$$
$$C:\ \mathbf{r}(t) = 2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$$
$$0 \le t \le \pi$$

**LINE INTEGRALS IN DIFFERENTIAL FORM**

If $\mathbf{F}$ is a vector field of the form $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$, and *C* is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then $\mathbf{F} \cdot d\mathbf{r}$ is often written as $Mdx + Ndy$.

*The parenthesis are often omitted.

Example 4: Evaluate the integral $\int_C (2x - y)\,dx + (x + 3y)\,dy$ along the path $C$

$C:$ arc on $y = x^{3/2}$ from $(0,0)$ to $(4,8)$

# Section 15.3 Conservative Vector Fields and Independence of Path

When you are done with your homework you should be able to…

- π Understand and use the Fundamental Theorem of Line Integrals
- π Understand the concept of independence of path
- π Understand the concept of conservation of energy

Warm-up: Show that the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ is the same for each parametric representation of *C*.

$\mathbf{F}(x,y)=(x^2+y^2)\mathbf{i}-x\mathbf{j}$

(a) $\mathbf{r}_1(t)=t\mathbf{i}+\sqrt{t}\mathbf{j},\ 0\le t\le 4$

(b) $\mathbf{r}_2(w)=w^2\mathbf{i}+w\mathbf{j},\ 0\le w\le 2$

**FUNDAMENTAL THEOREM OF LINE INTEGRALS**

Let $C$ be a piecewise smooth curve lying in an open region $R$ and given by $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j},\ a\le t\le b$. If $\mathbf{F}(x,y)=M\mathbf{i}+N\mathbf{j}$ is conservative in $R$, and $M$ and $N$ are continuous in $R$, then

where $f$ is a potential function of $\mathbf{F}$. That is, $\mathbf{F}(x,y)=\nabla f(x,y)$.

Example 1: Find the value of the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$.

$\mathbf{F}(x, y, z) = \mathbf{i} + z\mathbf{j} + y\mathbf{k}$

(a) $\mathbf{r}_1(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t^2\mathbf{k},\ 0 \le t \le \pi$

(b) $\mathbf{r}_2(t) = (1 - 2t)\mathbf{i} + \pi^2 t\mathbf{k},\ 0 \le t \le 1$

**THEOREM: INDEPENDENCE OF PATH AND CONSERVATIVE VECTOR FIELDS**

If $\mathbf{F}$ is continuous on an open connected region, then the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if and only if $\mathbf{F}$ is conservative.

**THEOREM: EQUIVALENT CONDITIONS**

Let $\mathbf{F}(x,y,z)=M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$ have continuous first partial derivatives in an open connected region $R$, and let $C$ be a piecewise smooth curve in $R$. The following conditions are equivalent:

1. $\mathbf{F}$ is conservative. That is, ______________ for some function $f$.
2. $\int_C \mathbf{F}\cdot d\mathbf{r}$ is ____________________ of path.
3. $\int_C \mathbf{F}\cdot d\mathbf{r}=0$ for every _________________ curve $C$ in $R$.

Example 2: Evaluate the line integral using the Fundamental Theorem of Line Integrals.

$$\int_C \left[2(x+y)\mathbf{i}+2(x+y)\mathbf{j}\right]\cdot d\mathbf{r}$$

$C$: smooth curve from $(-2,2)$ to $(4,3)$

Example 4: Evaluate the line integral using the Fundamental Theorem of Line Integrals.

$$\int_C \frac{2x}{\left(x^2+y^2\right)^2}\,dx + \frac{2y}{\left(x^2+y^2\right)^2}\,dy$$

$C$: circle $(x-4)^2+(y-5)^2=9$

clockwise from $(7,5)$ to $(1,5)$

# Section 15.4 Green's Theorem

When you are done with your homework you should be able to…

- π Use Green's Theorem to evaluate a line integral
- π Use alternative forms of Green's Theorem

Warm-up:

1. Represent the plane curve $2x-3y+5=0$ by a vector-valued function.

2. Determine whether the vector field $\mathbf{F}$ is conservative. If it is, find a potential function for the vector field.

$$\mathbf{F}(x,y,z)=\frac{x}{x^2+y^2}\mathbf{i}+\frac{y}{x^2+y^2}\mathbf{j}+\mathbf{k}$$

**SIMPLE CURVES:**

- A curve *C* given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad a \le t \le b$ is **simple** if it does not _______________ itself—that is $\mathbf{r}(c) \neq \mathbf{r}(d)$ for all *c* and *d* in the open interval $(a, b)$
- A plane region *R* is **simply connected** if its boundary consists of __________ simple _____________ curve

**GREEN'S THEOREM**

Let $R$ be a simply connected region with a piecewise smooth boundary $C$, oriented counterclockwise (that is, $C$ is traversed once so that the region $R$ always lies to the left). If $M$ and $N$ have continuous partial derivatives in an open region containing $R$, then

Example 1: Verify Green's Theorem by evaluating both integrals $\int_C y^2\,dx + x^2\,dy = \int_R\int\left(\frac{dN}{dx} - \frac{dM}{dy}\right)dA$ for the given path.

$C$: triangle with vertices $(0,0)$, $(4,0)$, $(4,4)$

Example 2: Use Green's Theorem to evaluate the integral $\int_C (y-x)\,dx + (2x-y)\,dy$ for the given path.

$C:\ x = 2\cos\theta,\ y = \sin\theta$

Example 3: Use Green's Theorem to evaluate the line integral.

$$\int_C \left(e^{-x^2/2} - y\right)dx + \left(e^{-y^2/2} + x\right)dy$$

$C:$ boundary of the region lying between
the graphs of the circle $x = 6\cos\theta,\ y = 6\sin\theta$
and the ellipse $x = 3\cos\theta,\ y = 2\sin\theta$

**THEOREM: LINE INTEGRAL FOR AREA**

If $R$ is a plane region bounded by a piecewise smooth simple closed curve $C$, oriented counterclockwise, then the area of $R$ is given by

Example 4: Use a line integral to find the area of the region $R$.

$R$: triangle bounded by the graphs of $x = 0,\ 3x - 2y = 0,\ x + 2y = 8$

**ALTERNATIVE FORMS OF GREEN'S THEOREM**

If $\mathbf{F}$ is a vector field in the plane, you can write $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + 0\mathbf{k}$. Thus, the

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ M & N & 0 \end{vmatrix} = -\frac{dN}{dz}\mathbf{i} + \frac{dM}{dz}\mathbf{j} + \left(\frac{dN}{dx} - \frac{dM}{dy}\right)\mathbf{k}$$

and $(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{dN}{dx} - \frac{dM}{dy}$. With appropriate conditions on $\mathbf{F}$, $C$ and $R$, you can write Green's Theorem in the vector form

(First alternative form)

Assume the same conditions for $\mathbf{F}$, $C$ and $R$. Using the arc length parameter $s$ for *C*, you have $\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$. So a unit tangent vector $\mathbf{T}$ to the curve *C* is given by $\mathbf{r}'(s) = x'(s)\mathbf{i} + y'(s)\mathbf{j}$ and the outward unit normal vector $\mathbf{N}$ can be written as $\mathbf{N} = y'(s)\mathbf{i} - x'(s)\mathbf{j}$. So for $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$ we have,

(Second alternative form)

# Section 15.5 Parametric Surfaces

When you are done with your homework you should be able to…

- $\pi$ Understand the definition of and sketch a parametric surface
- $\pi$ Find a set of parametric equations to represent a surface
- $\pi$ Find a normal vector and a tangent plane to a parametric surface
- $\pi$ Find the area of a parametric surface

Warm-up:

Find the unit tangent vector $\mathbf{T}(t)$ and find a set of parametric equations for the line tangent to the space curve $\mathbf{r}(t)=\langle 2\sin t, 2\cos t, 4\sin^2 t\rangle$ at the point $(1,\sqrt{3},1)$.

How do you represent a curve in the plane by a vector-valued function?

How do you represent a curve in space by a vector-valued function?

**DEFINITION OF PARAMETRIC SURFACE**

Let $x, y$ and $z$ be functions of $u$ and $v$ that are continuous on a domain *D* in the *uv*-plane. The set of points $(x, y, z)$ given by

is called a **parametric surface**. The equations

are the **parametric equations** for the surface.

Example 1: Find the rectangular equation for the surface by eliminating the parameters from the vector-valued function. Identify the surface and sketch its graph.

$$\mathbf{r}(u,v) = 2u\cos v\mathbf{i} + 2u\sin v\mathbf{j} + \frac{1}{2}u^2\mathbf{k}$$

Example 2: Find a vector-valued function whose graph is the indicated surface.

The plane $x + y + z = 6$

Example 3: Write a set of parametric equations for the surface of revolution obtained by revolving the graph of $y = x^{3/2},\ \ 0 \le x \le 4$ about the *x*-axis.

Example 4: Find an equation of the tangent plane to the surface represented by the vector-valued function $\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + \sqrt{uv}\,\mathbf{k}$ at the point $(1,1,1)$.

**AREA OF A PARAMETRIC SURFACE**

Let $S$ be a smooth parametric surface $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$ defined over an open region $D$ in the *uv*-plane. If each point on the surface $S$ corresponds to exactly one point in the domain $D$, then the **surface area** of $S$ is given by

Example 5: Find the area of the surface over the part of the paraboloid $\mathbf{r}(u,v) = 4u\cos v\mathbf{i} + 4u\sin v\mathbf{j} + u^2\mathbf{k}$, where $0 \le u \le 2$ and $0 \le v \le 2\pi$.

# Section 15.6 Surface Integrals

When you are done with your homework you should be able to…

- $\pi$ Evaluate a surface integral as a double integral
- $\pi$ Evaluate a surface integral for a parametric surface
- $\pi$ Determine the orientation of a surface
- $\pi$ Understand the concept of a flux integral

Warm-up: Find the principal unit normal vector to the curve $\mathbf{r}(t) = \ln t\,\mathbf{i} + (t+1)\,\mathbf{j}$ when $t = 2$.

**EVALUATING A SURFACE INTEGRAL**

Let $S$ a surface with equation $z = g(x, y)$ and let $R$ be its projection onto the *xy*-plane. If $g$, $g_x$, and $g_y$ are continuous on $R$ and $f$ is continuous on $S$, then the **surface integral of *f* over *S*** is

Example 1: Evaluate $\int_S\int (x-2y+z)\,dS$.

$$S:\ z=\frac{2}{3}x^{3/2},\quad 0\le x\le 1,\quad 0\le y\le x$$

Example 2: Evaluate $\int_S\int f(x,y)\,dS$.

$$f(x,y)=x+y$$
$$S:\ \mathbf{r}(u,v)=4u\cos v\mathbf{i}+4u\sin v\mathbf{j}+3u\mathbf{k}$$
$$0\le u\le 4,\quad 0\le v\le \pi$$

Example 3: Evaluate $\int_S\int f(x,y,z)\,dS$.

$$f(x,y,z)=\frac{xy}{z}$$
$$S:\ z=x^2+y^2,\quad 4\le x^2+y^2\le 16$$

**DEFINITION OF FLUX INTEGRAL**

Let $\mathbf{F}(x,y,z)=M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$ where $M$, $N$, and $P$ have continuous first partial derivatives on the surface $S$ oriented by a unit normal vector $\mathbf{N}$. The **flux integral** of $\mathbf{F}$ across $S$ is given by

**THEOREM: EVALUATING A FLUX INTEGRAL**

Let $S$ be an oriented surface given by $z = g(x, y)$ and let $R$ be its projection onto the *xy*-plane.

Example 4: Find the flux of $\mathbf{F}$ through $S$, $\int_S \int \mathbf{F} \cdot \mathbf{N}\, dS$, where $\mathbf{N}$ is the upward unit normal vector to $S$.

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j}$$
$$S:\ 2x + 3y + z = 6,\ \text{first octant}$$

# Section 15.7 Divergence Theorem

When you are done with your homework you should be able to…

- π Understand and use the Divergence Theorem
- π Use the Divergence Theorem to calculate flux

Warm-up: Find the flux of $\mathbf{F}$ through $S$, $\int_S\int \mathbf{F}\cdot\mathbf{N}dS$, where $\mathbf{N}$ is the upward unit normal vector to $S$.

$$\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$$
$$S:\ z = \sqrt{a^2 - x^2 - y^2}$$

**THEOREM: THE DIVERGENCE THEOREM (aka GAUSS'S THEOREM)**

Let $Q$ be a solid region bounded by a closed surface $S$ oriented by a unit normal vector directed outward from $Q$. If $\mathbf{F}$ is a vector field whose component functions have continuous partial derivatives in $Q$, then

Example 1: Verify the Divergence Theorem by evaluating $\int_S\int \mathbf{F}\cdot\mathbf{N}\,dS$ as a surface integral and as a triple integral.

$\mathbf{F}(x,y,z) = xy\mathbf{i} + z\mathbf{j} + (x+y)\mathbf{k}$

$S$ : surface bounded by the planes $y = 4$, and $z = 4 - x$ and the coordinate planes

Example 2: Use the Divergence Theorem to evaluate $\int_S\int \mathbf{F}\cdot\mathbf{N}\,dS$ and find the outward flux of $\mathbf{F}$ through the surface of the solid bounded by the graphs of the equations.

$\mathbf{F}(x,y,z) = xyz\mathbf{j}$

$S:\ x^2+y^2=9,\ z=0,\ z=4$

Example 3: Use the Divergence Theorem to evaluate $\int_S\int \mathbf{F}\cdot\mathbf{N}\,dS$ and find the outward flux of $\mathbf{F}$ through the surface of the solid bounded by the graphs of the equations.

$$\mathbf{F}(x,y,z) = 2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$
$$S:\ z = \sqrt{4 - x^2 - y^2},\ z = 0$$