When you are done with your homework you should be able to…

- π Understand and use the concept of a piecewise smooth curve
- π Write and evaluate a line integral
- π Write and evaluate a line integral of a vector field
- π Write and evaluate a line integral in differential form

Warm-up:

1. Represent the plane curve $2x-3y+5=0$ by a vector-valued function.

2. Determine whether the vector field $\mathbf{F}$ is conservative. If it is, find a potential function for the vector field.

$$\mathbf{F}(x,y,z)=\frac{x}{x^2+y^2}\mathbf{i}+\frac{y}{x^2+y^2}\mathbf{j}+\mathbf{k}$$

PIECEWISE SMOOTH CURVES:

- The work done by gravity on an object moving between two points in the field is independent of the path taken by the object
  - One constraint is that the **path** must be a piecewise smooth curve
- Recall that a plane curve *C* given by $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j},\quad a\le t\le b$ is smooth if $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous on $[a,b]$ and not simultaneously 0 on (a,b). Similarly, a space curve *C* given by $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k},\quad a\le t\le b$ is smooth if $\frac{dx}{dt}, \frac{dy}{dt}$ and $\frac{dz}{dt}$ are continuous on $[a,b]$ and not simultaneously 0 on (a,b).
- A curve *C* is **<u>piecewise smooth</u>** if the interval can be partitioned into a finite number of subintervals, on each of which *C* is smooth.

Example 1: Find a piecewise smooth parametrization of the path *C*.

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

DEFINITION OF LINE INTEGRAL

If f is defined in a region containing a smooth curve C of finite length, then the **line integral of *f* along *C*** is given by

$$\int_C f(x, y)\, ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta s_i \qquad \text{plane}$$

or

$$\int_C f(x, y, z)\, ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\, \Delta s_i \qquad \text{space}$$

provided this limit exists.

*To evaluate a line integral over a plane curve C given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, use the fact that $ds = \|\mathbf{r}'(t)\|\, dt = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$.

THEOREM: EVALUATION OF A LINE INTEGRAL AS A DEFINITE INTEGRAL

Let f be continuous in a region containing a smooth curve C.

If C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, where $a \le t \le b$, then

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

If C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where $a \le t \le b$, then

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

Note that if $f(x, y, z) = 1$, the line integral gives the arc length of the curve C.

That is, $\int_C 1\, ds = \int_a^b \|r'(t)\|\, dt = \text{length of curve } C$.

Example 2: Evaluate the line integral along the given path.

$$\int_C 8xyz\,ds$$

$$C: \ \mathbf{r}(t) = 12t\mathbf{i} + 5t\mathbf{j} + 3\mathbf{k}.$$

$$0 \le t \le 2$$

DEFINITION OF LINE INTEGRAL OF A VECTOR FIELD

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve *C* given by $\mathbf{r}(t)$, $a \le t \le b$. The **line integral of F on *C*** is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\,ds = \int_a^b \mathbf{F}\left(x(t), y(t), z(t)\right) \cdot \mathbf{r}'(t)\,dt$$

Example 3: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is represented by $\mathbf{r}(t)$

$$\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$$

$$C: \ \mathbf{r}(t) = 2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$$

$$0 \le t \le \pi$$

LINE INTEGRALS IN DIFFERENTIAL FORM

If $\mathbf{F}$ is a vector field of the form $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$, and C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then $\mathbf{F}\cdot d\mathbf{r}$ is often written as $Mdx + Ndy$.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt \\ &= \int_a^b (M\mathbf{i}+N\mathbf{j})\cdot\left(x'(t)\mathbf{i}+y'(t)\mathbf{j}\right)dt \\ &= \int_a^b \left(M\frac{dx}{dt}+N\frac{dy}{dt}\right)dt \\ &= \int_C (Mdx+ndy)\end{aligned}$$

*The parenthesis are often omitted.

Example 4: Evaluate the integral $\int_C (2x-y)\,dx+(x+3y)\,dy$ along the path C

$$C:\ \text{arc on } y = x^{3/2} \text{ from } (0,0) \text{ to } (4,8)$$