When you are done with your homework you should be able to…

- π Understand the concept of a vector field
- π Determine whether a vector field is conservative
- π Find the curl of a vector field
- π Find the divergence of a vector field

Warm-up: A 48,000-pound truck is parked on a 10° slope. Assume the only force to overcome is that due to gravity.

a. Find the force required to keep the truck from rolling down the hill.

b. Find the force perpendicular to the hill.

DEFINITION OF VECTOR FIELD

Let M and N be functions of two variables x and y, defined on a plane region R. The function $\mathbf{F}$ defined by $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is called a **vector field over *R***. (plane)

Let M, N, and P be functions of three variables x, y and z, defined on a solid region Q. The function $\mathbf{F}$ defined by $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is called a **vector field over *Q***. (space)

Example 1: Sketch several representative vectors in the vector field $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$.

Example 2: Sketch several representative vectors in the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

DEFINITION OF INVERSE SQUARE FIELD

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be a position vector. The vector field $\mathbf{F}$ is an **inverse square field** if

$$\mathbf{F}(x, y, z) = \frac{k}{\|\mathbf{r}\|^2}\mathbf{u}$$

where k is a real number and $\mathbf{u} = \frac{\mathbf{r}}{\|\mathbf{r}\|}$ is a unit vector in the direction of $\mathbf{r}$.

DEFINITION OF CONSERVATIVE VECTOR FIELD

A vector field $\mathbf{F}$ is called **conservative** if there exists a differentiable function f such that $\mathbf{F} = \nabla f$. The function f is called the **potential function** for $\mathbf{F}$.

Example 3: Find the gradient vector field for the scalar function. That is, find the conservative vector field for the potential function.

$$f(x, y, z) = \frac{y}{z} + \frac{z}{x} - \frac{xz}{y}$$

THEOREM: TEST FOR CONSERVATIVE VECTOR FIELD IN THE PLANE

Let M and N have continuous first partial derivatives on an open disk R. The vector field given by $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is conservative if and only if $\frac{dN}{dx} = \frac{dM}{dy}$.

Example 4: Determine whether the vector field is conservative. If it is, find a potential function for the vector field.

$$\mathbf{F}(x, y) = \frac{1}{y^2}(y\mathbf{i} - 2x\mathbf{j})$$

DEFINITION OF A CURL OF A VECTOR FIELD

The curl of $\mathbf{F}(x,y,z)=M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$ is

$$\begin{aligned}\textbf{curl }\mathbf{F}(x,y,z)&=\nabla\times\mathbf{F}(x,y,z)\\&=\left(\frac{dP}{dy}-\frac{dN}{dz}\right)\mathbf{i}-\left(\frac{dP}{dx}-\frac{dM}{dz}\right)\mathbf{j}+\left(\frac{dN}{dx}-\frac{dM}{dy}\right)\mathbf{k}\end{aligned}$$

Example 5: Find curl $\mathbf{F}$ for the vector field $\mathbf{F}(x,y,z)=e^{-xyz}(\mathbf{i}+\mathbf{j}+\mathbf{k})$ at the point $(3,2,0)$.

THEOREM: TEST FOR CONSERVATIVE VECTOR FIELD IN SPACE

Suppose that M, N and P have continuous first partial derivatives on an open sphere Q in space. The vector field given by $\mathbf{F}(x,y,z)=M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$ is conservative if and only if

$$\textbf{curl }\mathbf{F}(x,y,z)=\mathbf{0}.$$

That is, $\mathbf{F}$ is conservative if and only if

$$\frac{dP}{dy}=\frac{dN}{dz},\ \frac{dP}{dx}=\frac{dM}{dz},\ \text{and }\frac{dN}{dx}=\frac{dM}{dy}.$$

Example 6: Determine whether the vector field is conservative. If it is, find a potential function for the vector field.

$$\mathbf{F}(x, y, z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$$

DEFINITION: DIVERGENCE OF A VECTOR FIELD

The **divergence** of $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is

$$\begin{aligned}\operatorname{div}\mathbf{F}(x, y) &= \nabla \cdot \mathbf{F}(x, y)\\ &= \frac{dM}{dx} + \frac{dN}{dy}.\end{aligned}$$

The **divergence** of $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ $\mathbf{F}$ is

$$\begin{aligned}\operatorname{div}\mathbf{F}(x, y, z) &= \nabla \cdot \mathbf{F}(x, y, z)\\ &= \frac{dM}{dx} + \frac{dN}{dy} + \frac{dP}{dz}.\end{aligned}$$

If $\operatorname{div}\mathbf{F}(x, y, z) = 0$, then $\mathbf{F}$ is said to be **divergence free**.

Example 7: Find the divergence of the vector field $\mathbf{F}(x, y, z) = \ln(xyz)(\mathbf{i} + \mathbf{j} + \mathbf{k})$ at the point $(3, 2, 1)$.

THEOREM: RELATIONSHIP BETWEEN DIVERGENCE AND CURL

If $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field and M, N and P have continuous second partial derivatives, then

$$\operatorname{div}(\mathbf{curl\ F}) = 0.$$