When you are done with your homework you should be able to…

- π Find and use directional derivatives of a function of two variables
- π Find the gradient of a function of two variables
- π Use the gradient of a function of two variables in applications
- π Find directional derivatives and gradients of functions of three variables

Warm-up: Normalize the following vector (aka find the unit vector):

$$\mathbf{v} = 6\mathbf{i} - \mathbf{j}$$

Recall that the slope of a surface in the *x*-direction is given by ________________

And the slope of a surface in the *y*-direction is given by ________________.

In this section, we will find that these two __________________ ______________

can be used to find the slope in any direction.

DEFINITION: DIRECTIONAL DERIVATIVE

Let f be a function of two variables x and y and let $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ be a unit vector. Then the **directional derivative of f in the direction of u**, denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x, y) = \lim_{t\to 0}\frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t}$$

provided the limit exists.

THEOREM: DIRECTIONAL DERIVATIVE

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u} = \cos\theta \mathbf{i} + \sin\theta \mathbf{j}$ is

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

There are infinitely many directional derivatives to a surface at a given point—one for each direction specified by $\mathbf{u}$.

Example 1: Find the directional derivative of the following functions at the given point and direction.

a. $f(x, y) = x^3 - y^3$, at the point $P(4,3)$, in the direction $\mathbf{v} = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$

b. $f(x, y) = \cos(x + y)$, at the point $P(0, \pi)$, in the direction $Q\left(\frac{\pi}{2}, 0\right)$

DEFINITION: GRADIENT OF A FUNCTION OF TWO VARIABLES

Let $z = f(x, y)$, be a function of x and y such that f_x and f_y exist. Then the **gradient of *f***, denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

∇f is read as "del *f*". Another notation for the gradient is $\mathbf{grad}\, f(x, y)$.

Example 2: Find the gradient of $f(x, y) = \ln(x^2 - y)$, at the point $(2, 3)$.

THEOREM: ALTERNATIVE FORM OF THE DIRECTIONAL DERIVATIVE

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Example 3: Use the gradient to find the directional derivative of the function $f(x, y) = \sin 2x \cos y$ at the point $P(0, 0)$ in the direction of $Q\left(\frac{\pi}{2}, \pi\right)$.

THEOREM: PROPERTIES OF THE GRADIENT

Let f be differentiable at the point (x, y).

1. If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y) = 0$ for all $\mathbf{u}$
2. The direction of *maximum* increase of f is given by $\nabla f(x, y)$. The maximum value of $D_{\mathbf{u}} f(x, y)$ is $\|\nabla f(x, y)\|$.
3. The direction of *minimum* increase of f is given by $-\nabla f(x, y)$. The minimum value of $D_{\mathbf{u}} f(x, y)$ is $-\|\nabla f(x, y)\|$.

Example 4: The surface of a mountain is modeled by the equation $h(x, y) = 5000 - 0.001x^2 - 0.004y^2$. A mountain climber is at the point $(500, 300, 4390)$. In what direction should the climber move in order to ascend at the greatest rate?

THEOREM: GRADIENT IS NORMAL TO LEVEL CURVES

If f is differentiable at (x_0, y_0) and $\nabla f(x, y) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).

THEOREM: PROPERTIES OF THE GRADIENT

Let f be a function of , with continuous first partial derivatives. The **directional derivative of f** in the direction of a unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is given by

$$D_{\mathbf{u}} f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf(x, y, z)$$

The **gradient of f** is defined to be

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

1. $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y, z) = 0$ for all $\mathbf{u}$
3. The direction of *maximum* increase of f is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}} f(x, y, z)$ is $\|\nabla f(x, y, z)\|$.
4. The direction of *minimum* increase of f is given by $-\nabla f(x, y, z)$. The minimum value of $D_{\mathbf{u}} f(x, y, z)$ is $-\|\nabla f(x, y, z)\|$.

Example 5: Find the gradient of the function $w = xy^2z^2$ and the maximum value of the directional derivative at the point $(2,1,1)$.