When you are done with your homework you should be able to…

- π Use the chain rules for functions of several variables
- π Find partial derivatives implicitly

Warm-up: A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is eight feet deep.

THEOREM: CHAIN RULE: ONE INDEPENDENT VARIABLE

Let $w = f(x, y)$, where f is a differentiable function x and y. If $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt} + \frac{dw}{dy}\frac{dy}{dt}$$

This can be extended to any number of variables. If $w = f(x_1, x_2, ..., x_n)$, you would have

$$\frac{dw}{dt} = \frac{dw}{dx_1}\frac{dx_1}{dt} + \frac{dw}{dx_2}\frac{dx_2}{dt} + \cdots + \frac{dw}{dx_n}\frac{dx_n}{dt}$$

Example 1: Find $\frac{dw}{dt}$ (a) using the appropriate chain rule and (b) by converting w to a function of t before differentiating.

a. $w=\cos(x-y),\ \ x=t^2,\ \ y=1$

b. $w=xyz,\ \ x=t^2,\ \ y=2t,\ \ z=e^{-t}$

THEOREM: CHAIN RULE: ONE INDEPENDENT VARIABLE

Let $w=f(x,y)$, where f is a differentiable function x and y. If $x=g(s,t)$ and $y=h(s,t)$, such that the first partials $dx/ds,\ dx/dt,\ dy/ds,$ and dy/dt all exist, then $\frac{dw}{ds}$ and $\frac{dw}{dt}$ exist and are given by

$$\frac{dw}{ds}=\frac{dw}{dx}\frac{dx}{ds}+\frac{dw}{dy}\frac{dy}{ds} \text{ and } \frac{dw}{dt}=\frac{dw}{dx}\frac{dx}{dt}+\frac{dw}{dy}\frac{dy}{dt}$$

This can be extended to any number of variables. If w is a differentiable function of the n variables where each $x_1,x_2,\ldots,x_n$ is a differentiable function of the m variables $t_1,t_2,\ldots,t_m$, then for $w=f(x_1,x_2,\ldots,x_n)$, you would have

$$\frac{dw}{dt_1}=\frac{dw}{dx_1}\frac{dx_1}{dt_1}+\frac{dw}{dx_2}\frac{dx_2}{dt_1}+\cdots+\frac{dw}{dx_n}\frac{dx_n}{dt_1}$$

$$\frac{dw}{dt_2}=\frac{dw}{dx_1}\frac{dx_1}{dt_2}+\frac{dw}{dx_2}\frac{dx_2}{dt_2}+\cdots+\frac{dw}{dx_n}\frac{dx_n}{dt_2}$$

$$\vdots$$

$$\frac{dw}{dt_m}=\frac{dw}{dx_1}\frac{dx_1}{dt_m}+\frac{dw}{dx_2}\frac{dx_2}{dt_m}+\cdots+\frac{dw}{dx_n}\frac{dx_n}{dt_m}$$

Example 2: Find dw/ds and dw/dt using the appropriate chain rule, and evaluate each partial derivative at the given values of *s* and *t*.

<u>Function</u>	<u>Point</u>
$w=y^3-3x^2y$	$s=0,\quad t=1$
$x=e^s,\quad y=e^t$	

THEOREM: CHAIN RULE: IMPLICIT DIFFERENTIATION

If the equation $F(x,y)=0$ defines y implicitly as a differentiable function of x, then

$$\frac{dy}{dx}=-\frac{F_x(x,y)}{F_y(x,y)},\quad F_y(x,y)\neq 0.$$

If the equation $F(x,y,z)=0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{dz}{dx}=-\frac{F_x(x,y,z)}{F_z(x,y,z)} \text{ and } \frac{dz}{dy}=-\frac{F_y(x,y,z)}{F_z(x,y,z)},\quad F_z(x,y,z)\neq 0.$$

Example 3: Differentiate implicitly to find $\frac{dy}{dx}$.

$$\cos x+\tan xy+5=0$$

Example 4: Differentiate implicitly to find the first partial derivatives of z.

$$x \ln y + y^2 z + z^2 = 8$$

Example 5: The radius of a right circular cone is increasing at a rate of 6 inches per minute, and the height is decreasing at a rate of 4 inches per minute. What are the rates of change of the volume and surface area when the radius is 12 inches and the height is 36 inches?