When you are done with your homework you should be able to…

- π Find the arc length of a space curve
- π Use the arc length parameter to describe a plane curve or space curve
- π Find the curvature of a curve at a point on the curve
- π Use a vector-valued function to find frictional force

Warm-up: Find the arc length of the curve
$x = \arcsin t$ and $y = \ln\sqrt{1-t^2}$ on the interval $\left[0, \frac{1}{2}\right]$.

THEOREM: ARC LENGTH OF A SPACE CURVE

If C is a smooth curve given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ on an interval $[a,b]$, then the **arc length of C** is

$$s = \int_a^b \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2 + \left[z'(t)\right]^2}\,dt = \int_a^b \left\|\mathbf{r}'(t)\right\|\,dt$$

Example 1: Find the arc length of the curve given by $\mathbf{r}(t) = 2\sin t\mathbf{i} + 5t\mathbf{j} + 2\cos t\mathbf{k}$ over $[0, \pi]$.

DEFINITION: ARC LENGTH FUNCTION

Let *C* be a smooth curve given by $\mathbf{r}(t)$ defined on the closed interval $[a, b]$, then the **arc length of *C*** is

$$s(t) = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\, du = \int_a^t \|\mathbf{r}'(u)\|\, du$$

The arc length *s* is called the **arc length parameter**. The arc length function is nonnegative as it measures the distance along *C* from the initial point. Using the definition of the arc length function and the second fundamental theorem of calculus, you can conclude $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$.

Example 2: Find the arc length function for the line segment given by $\mathbf{r}(t) = (3 - 3t)\mathbf{i} + 4t\mathbf{j},\ 0 \le t \le 1$.and write $\mathbf{r}$ as a function of the parameter *s.*

THEOREM: ARC LENGTH PARAMETER

If C is a smooth curve given by

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} \text{ or } \mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k}$$

where s is the **arc length parameter**, then

$$\|\mathbf{r}'(s)\| = 1.$$

Moreover, if t is any parameter for the vector-valued function $\mathbf{r}$ such that $\|\mathbf{r}'(s)\| = 1$, then t must be the arc length parameter.

DEFINITION OF CURVATURE

Let C be a smooth curve (in the plane or in space) given by $\mathbf{r}(s)$, where s is the arc length parameter. The **curvature** K at s is given by

$$K = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{T}'(s)\|$$

Example 3: Find the curvature K of the curve, where s is the arc length parameter.

$$\mathbf{r}(s) = (3+s)\mathbf{i} + \mathbf{j}$$

THEOREM: FORMULAS FOR CURVATURE

If C is a smooth curve given by $\mathbf{r}(t)$, then the curvature K of C at t is given by

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

Example 4: Find the curvature K of the curve $\mathbf{r}(t) = 2t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$.

THEOREM: CURVATURE IN RECTANGULAR COORDINATES

If C is the graph of a twice differentiable function given by $y = f(x)$, then the curvature K at the point (x, y) is given by

$$K = \frac{|y''|}{\left[1+(y')^2\right]^{3/2}}$$

Related Stuff: Let C be a curve with curvature K at point P. The circle passing through point P with radius $r = \frac{1}{K}$ is called the **circle of curvature** if the circle lies on the concave side of the curve and shares a common tangent line with the curve at point P. The radius is called the **radius of curvature** at P and the center of the circle is called the **center of curvature**.

Example 4: Find the curvature and radius of curvature of the plane curve $y = 2x + \frac{4}{x}$ at $x = 1$.

THEOREM: ACCELERATION, SPEED, AND CURVATURE

If $\mathbf{r}(t)$ is the position vector for a smooth curve C then the acceleration vector is given by

$$\mathbf{a}(t) = \frac{d^2 s}{dt^2}\mathbf{T} + K\left(\frac{ds}{dt}\right)^2 \mathbf{N}$$

where K is the curvature of C and $\frac{ds}{dt}$ is the speed.

<u>Frictional Force</u>

A moving object with mass m is in contact with a stationary object. The total force required to produce an acceleration $\mathbf{a}$ along a given path is

$$\begin{aligned}\mathbf{F} &= m\mathbf{a} \\ &= m\left(\frac{d^2 s}{dt^2}\right)\mathbf{T} + mK\left(\frac{ds}{dt}\right)^2 \mathbf{N} \\ &= ma_{\mathbf{T}}\mathbf{T} + ma_{\mathbf{N}}\mathbf{N}\end{aligned}$$

Example 5: A 6400-pound vehicle is driven at a speed of 35 mph on a circular interchange of radius 250 feet. To keep the vehicle from skidding off course, what frictional force must the road surface exert on the tires?