When you are done with your homework you should be able to…

- π Use properties of the dot product of two vectors
- π Find the angle between two vectors using the dot product
- π Find the direction cosines of a vector in space
- π Find the projection of a vector onto another vector
- π Use vectors to find the work done by a constant force

Warm-up: Write the equation of the sphere in standard form. Find the center and the radius

$$9x^2 + 9y^2 + 9z^2 - 6x + 18y + 1 = 0$$

DEFINITION OF DOT PRODUCT (aka inner product aka scalar product)

The dot product of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

The dot product of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

THEOREM: PROPERTIES OF THE DOT PRODUCT

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in the plane or in space and let c be a scalar.

1. *Commutative Property.* $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$
2. *Distributive Property.* $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$
3. $c(\mathbf{u}\cdot\mathbf{v})=c\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot c\mathbf{v}$
4. $\mathbf{0}\cdot\mathbf{v}=0$
5. $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2$

Example 1: Given $\mathbf{u}=\langle -4,6\rangle$, $\mathbf{v}=\langle 3,7\rangle$ and $\mathbf{w}=\langle 9,-5\rangle$, find each of the following:

a) $\mathbf{u}\cdot\mathbf{w}$

b) $5\mathbf{u}\cdot\mathbf{v}$

c) $\mathbf{u}\cdot\mathbf{u}$

d) $(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w}$

THEOREM: ANGLE BETWEEN TWO VECTORS

If $\theta,\ 0 \le \theta \le \pi,$ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}.$$

Example 2: Find the angle θ between the vectors $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}.$.

DEFINITION: ORTHOGONAL VECTORS

The vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if

$$\mathbf{u}\cdot\mathbf{v}=0.$$

Example 3: Determine whether vectors $\mathbf{u}=-2\mathbf{i}+3\mathbf{j}-\mathbf{k}$ and $\mathbf{v}=2\mathbf{i}+\mathbf{j}-\mathbf{k}$ are orthogonal, parallel or neither.

DIRECTION COSINES

For a vector in the *plane*, we often measure its direction in terms of the

_____________ measured ___________________ from the _______________

_____________ to the _______________.

In *space*, it is more convenient to measure direction in terms of the angles between the nonzero vector $\mathbf{v}$ and the three unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. The angles α, β and γ are the **direction angles** of $\mathbf{v}$ and $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the **direction cosines** of $\mathbf{v}$.

Activity:

1. Use the theorem for the angle between two vectors to find an alternate form of the dot product. Substitute the unit vector $\mathbf{i}$ for vector $\mathbf{u}$.

2. Now find $\mathbf{v}\cdot\mathbf{i}$ using the component form of each vector.

3. Equate your results from parts 1 and 2 and then isolate $\cos\alpha$.

4. Repeat this exercise to find $\cos\beta$ and $\cos\gamma$.

5. Find the normalized form of any nonzero vector $\mathbf{v}$, that is, find two expressions for $\frac{\mathbf{v}}{\|\mathbf{v}\|}$, using your previous results.

6. Find $\cos^2\alpha+\cos^2\beta+\cos^2\gamma$. Hint: $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector.

Example 4: Find the direction angles of the vector $\mathbf{u} = -4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$.

DEFINITION OF PROJECTION AND VECTOR COMPONENTS

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors and let $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to $\mathbf{v}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$.

1. $\mathbf{w}_1$ is called the projection of $\mathbf{u}$ onto $\mathbf{v}$ or the vector component of $\mathbf{u}$ along $\mathbf{v}$, and is denoted by $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$.
2. $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$ is called the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$.

THEOREM: PROJECTION USING THE DOT PRODUCT

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, then the projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

DEFINITION OF WORK

The work W done by a constant force $\mathbf{F}$ as its point of application moves along the vector $\overrightarrow{PQ}$ is given by either of the following:

1. $W = \left\|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\right\|\left\|\overrightarrow{PQ}\right\|$
2. $W = \mathbf{F}\cdot\overrightarrow{PQ}$

Example 5: A toy wagon is pulled by exerting a force of 25 pounds on a handle that makes a 20° angle with the horizontal. Find the work done in pulling the wagon 50 feet.