When you are done with your homework you should be able to…

- π Differentiate a vector-valued function
- π Integrate a vector-valued function

Warm-up 1: Evaluate the following derivatives with respect to *x*.

1. $y=\frac{\sin^2 3x}{\sqrt{x}}$

2. $f(x)=xe^{-2x}$

3. $y=\ln\left(\frac{5x}{e^{x^2}}\right)^{2/3}-\frac{6}{x}-\arctan 3x^3$

Warm-up 2: Integrate.

1. $\int\left(6x^2-\sin^2 3x\right)dx$

2. $\int \frac{\sqrt{\ln x}}{x}\,dx$

3. $\int \frac{4}{\sqrt{1-x^2}}\,dx$

DEFINITION OF THE DERIVATIVE OF A VECTOR-VALUED FUNCTION

The **derivative of a vector-valued function $\mathbf{r}$** is defined by

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

for all t for which the limit exists. If $\mathbf{r}'(c)$ exists, then $\mathbf{r}$ is **differentiable at c**. If $\mathbf{r}'(c)$ exists for all c in an open interval I then $\mathbf{r}$ is **differentiable on the open interval I**. Differentiability of vector-valued functions can be extended to closed intervals by considering one-sided limits.

Other notation: $\mathbf{r}'(t),\ \frac{d}{dt}[\mathbf{r}(t)],\ D_t[\mathbf{r}(t)],\ \frac{d\mathbf{r}}{dt}$

THEOREM: DIFFERENTIATION OF VECTOR-VALUED FUNCTIONS

1. If $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}$, where f and g are differentiable functions of *t*, then $\mathbf{r}'(t)=f'(t)\mathbf{i}+g'(t)\mathbf{j}$. provided *f* and *g* have limits as $t \to a$.

2. If $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$, where f, g, and h are differentiable functions of *t*, then $\mathbf{r}'(t)=f'(t)\mathbf{i}+g'(t)\mathbf{j}+h'(t)\mathbf{k}$.

Higher-order derivatives of vector-valued functions are obtained by successive differentiation of each component function.

The **parametrization of the curve** represented by the vector-valued function $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$ is **smooth on an open interval *I*** if f', g', and h' are continuous on *I* and $\mathbf{r}'(t)\neq\mathbf{0}$ for any value of *t* on the open interval *I*.

THEOREM: PROPERTIES OF THE DERIVATIVE

Let $\mathbf{r}$ and $\mathbf{u}$ be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

1. $D_t\left[c\mathbf{r}(t)\right]=c\mathbf{r}'(t)$
2. $D_t\left[\mathbf{r}(t)\pm\mathbf{u}(t)\right]=\mathbf{r}'(t)\pm\mathbf{u}'(t)$
3. $D_t\left[f(t)\mathbf{u}(t)\right]=f(t)\mathbf{r}'(t)+f'(t)\mathbf{r}(t)$
4. $D_t\left[\mathbf{r}(t)\cdot\mathbf{u}(t)\right]=\mathbf{r}(t)\cdot\mathbf{u}'(t)+\mathbf{r}'(t)\cdot\mathbf{u}(t)$
5. $D_t\left[\mathbf{r}(t)\times\mathbf{u}(t)\right]=\mathbf{r}(t)\times\mathbf{u}'(t)+\mathbf{r}'(t)\times\mathbf{u}(t)$
6. $D_t\left[\mathbf{r}(f(t))\right]=\mathbf{r}'(f(t))\cdot f'(t)$
7. If $\mathbf{r}(t)\cdot\mathbf{r}(t)=c,$ then $\mathbf{r}(t)\cdot\mathbf{r}'(t)=0$

Example 1: Find $\mathbf{r}'(t)\cdot\mathbf{r}''(t)$.

$$\mathbf{r}(t)=(t^2+t)\mathbf{i}+(t^2-t)\mathbf{j}$$

Example 2: Find $D_t\left[\mathbf{r}(t)\times\mathbf{u}(t)\right]$

$\mathbf{r}(t)=t\mathbf{i}+2\sin t\mathbf{j}+2\cos t\mathbf{k},$

$\mathbf{u}(t)=\frac{1}{t}\mathbf{i}+2\sin t\mathbf{j}+2\cos t\mathbf{k},$

DEFINITION OF INTEGRATION OF VECTOR-VALUED FUNCTIONS

1. If $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}$, where f and g are continuous on $[a,b]$ then the <u>**indefinite integral (antiderivative)**</u> of $\mathbf{r}$ is

$$\int\mathbf{r}(t)\,dt=\left[\int f(t)\,dt\right]\mathbf{i}+\left[\int g(t)\,dt\right]\mathbf{j}$$

and its <u>**definite integral**</u> over the interval $a\le t\le b$ is

$$\int_a^b\mathbf{r}(t)\,dt=\left[\int_a^b f(t)\,dt\right]\mathbf{i}+\left[\int_a^b g(t)\,dt\right]\mathbf{j}$$

2. If $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$, where f, g, and h are continuous on $[a,b]$ then the <u>**indefinite integral (antiderivative)**</u> of $\mathbf{r}$ is

$$\int\mathbf{r}(t)\,dt=\left[\int f(t)\,dt\right]\mathbf{i}+\left[\int g(t)\,dt\right]\mathbf{j}+\left[\int h(t)\,dt\right]\mathbf{k}$$

and its <u>**definite integral**</u> over the interval $a\le t\le b$ is

$$\int_a^b\mathbf{r}(t)\,dt=\left[\int_a^b f(t)\,dt\right]\mathbf{i}+\left[\int_a^b g(t)\,dt\right]\mathbf{j}+\left[\int_a^b g(t)\,dt\right]\mathbf{k}$$

Example 3: Evaluate the indefinite integral

$$\int\left(4t^3\mathbf{i}+6t\mathbf{j}-4\sqrt{t}\mathbf{k}\right)dt$$

Example 4: Evaluate the definite integral

$$\int_0^{\pi/4}\left[\sec t\tan t\mathbf{i}+\tan t\mathbf{j}+2\sin t\cos t\mathbf{k}\right]dt$$