EXAM 3/CHAPTER 13.1-13.9
100 POINTS POSSIBLE

NAME Key

LEAVE ALL ANSWERS EXACT UNLESS THE PROBLEM INDICATES OTHERWISE
SHOW ALL WORK IN ORDER TO EARN FULL CREDIT

1. (10 POINTS) Consider the function $g(x,y)=\dfrac{xy}{x-y}$ at the point $(2,2,-1)$.

   a. (5 POINTS) Find the slope of the surface in the x-direction at the given point.

$$g_x = \frac{y(x-y)-xy(1)}{(x-y)^2} = \frac{-y^2}{(x-y)^2}$$

$\boxed{\text{at } g(2,2) \text{ the slope is undefined}}$

   b. (5 POINTS) Find the slope of the surface in the y-direction at the given point.

$$g_y = \frac{x(x-y)-xy(-1)}{(x-y)^2}$$

$$g_y = \frac{x^2}{(x-y)^2}$$

$\boxed{\text{at } g(2,2) \text{ the slope is undefined}}$

2. (20 POINTS) A triangle is measured and two adjacent sides are found to be 5 inches and 6 inches long, with an included angle of $\dfrac{\pi}{6}$. The possible errors in measurement are $\dfrac{1}{12}$ inch for the sides and $\dfrac{1}{50}$ radian for the angle. Find the maximum possible error in the computation of the area.

$a=5 \quad \partial a = \pm\frac{1}{12}$
$b=6 \quad \partial b = \pm\frac{1}{12}$
$C=\frac{\pi}{6} \quad \partial C = \pm\frac{1}{50}$

$$A = \tfrac{1}{2}ab\sin C$$

$$\partial A = \frac{1}{2}\left[b\sin C\,\partial a + a\sin C\,\partial b + ab\cos C\,\partial C\right]$$

$$\partial A = \frac{1}{2}\left[6\sin\tfrac{\pi}{6}\left(\tfrac{1}{12}\right) + 5\sin\tfrac{\pi}{6}\left(\tfrac{1}{12}\right) + 5\cdot 6\cos\tfrac{\pi}{6}\left(\tfrac{1}{50}\right)\right]$$

$$\partial A = \frac{1}{2}\left[\frac{1}{4} + \frac{5}{24} + \frac{3\sqrt{3}}{10}\right]$$

$$\partial A = \frac{1}{2}\left[\frac{60+50+72\sqrt{3}}{240}\right]$$

$$\partial A = \frac{1}{2}\left(\frac{110+72\sqrt{3}}{240}\right)$$

$$\boxed{\partial A = \pm\frac{55+36\sqrt{3}}{240}\ m^2}$$

3. (8 POINTS) Find the angle of inclination θ of the tangent plane to the surface $x^2+y^2=5$ at the point $(2,1,3)$. Let $F(x,y,z)=x^2+y^2-5$

$$\nabla F(x,y,z) = 2x\hat{i} + 2y\hat{j}$$

$$\nabla F(2,1,3) = 2(2)\hat{i} + 2(1)\hat{j}$$

$$\nabla F(2,1,3) = 4\hat{i} + 2\hat{j}$$

$$\cos\theta = \frac{|\nabla F(2,1,3)\cdot\hat{k}|}{\|\nabla F(2,1,3)\|}$$

$$\cos\theta = \frac{|2\cdot 0 + 1\cdot 0 + 0\cdot 0|}{\sqrt{4^2+2^2+0^2}}$$

$\cos\theta = 0$

$\boxed{\theta = 90^\circ \text{ or } \frac{\pi}{2}}$

4. (10 POINTS) Consider the function $h(x,y) = y\cos(x-y)$ at the point $\left(0,\frac{\pi}{3}\right)$.

a. (6 POINTS) Find the gradient of the function at the given point .

$$\nabla h(x,y) = -y\sin(x-y)\hat{i} + \left(\cos(x-y) - y\sin(x-y)(-1)\right)\hat{j}$$

$$\nabla h(x,y) = -y\sin(x-y)\hat{i} + \left(\cos(x-y) + y\sin(x-y)\right)\hat{j}$$

$$\nabla h(0,\pi/3) = -\pi/3\,\sin(-\pi/3)\hat{i} + \left(\cos(-\pi/3) + \pi/3\,\sin(-\pi/3)\right)\hat{j}$$

$$\boxed{\nabla h(0,\pi/3) = \frac{\pi\sqrt{3}}{6}\hat{i} + \left(\frac{3-\pi\sqrt{3}}{6}\right)\hat{j}}$$

b. (4 POINTS) Find the maximum value of the directional derivative of the function at the given point.

max value $= \|\nabla h(0,\pi/3)\|$

$$= \sqrt{\left(\frac{\pi\sqrt{3}}{6}\right)^2 + \left(\frac{1-\pi\sqrt{3}}{2}\right)^2} = \sqrt{\frac{3\pi^2 + 9 - 6\pi\sqrt{3} + 3\pi^2}{36}}$$

$$= \sqrt{\frac{3\pi^2}{36} + \frac{9 - 6\pi\sqrt{3} + 3\pi^2}{36}}$$

$$= \boxed{\frac{\sqrt{6\pi^2 - 6\pi\sqrt{3} + 9}}{6}}$$

5. (12 POINTS) Consider the surface $y\ln xz^2 = 2$ at the point $(e,2,1)$.

a. (8 POINTS) Find an equation of the tangent plane to the surface at the given point.

Let $F(x,y,z) = y\ln xz^2 - 2$

$$F_x(x,y,z) = y\left(\frac{\cancel{z^2}}{x\cancel{z^2}}\right) = \frac{y}{x} \to F_x(e,2,1) = \frac{2}{e} = a$$

$$F_y(x,y,z) = \ln xz^2 \to F_y(e,2,1) = \ln e = 1 = b$$

$$F_z(x,y,z) = y\left(\frac{2\cancel{x}\cancel{z}}{\cancel{x}z^{\cancel{2}}}\right) = \frac{2y}{z} \to F_z(e,2,1) = 4 = c$$

$$\boxed{\frac{2}{e}(x-e) + (y-2) + 4(z-1) = 0}$$

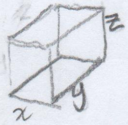
b. (4 POINTS) Find symmetric equations of the normal line to the surface at the given point.

$$\frac{x-e}{\frac{2}{e}} = \frac{y-2}{1} = \frac{z-1}{4}$$

$$\boxed{\frac{e(x-e)}{2} = y-2 = \frac{z-1}{4}}$$

6. (20 POINTS) A home improvement contractor is painting the walls and ceiling of a rectangular room. The volume of the room is 668.25 cubic feet. The cost of wall paint is \$0.06 per square foot and the cost of ceiling paint is \$0.11 per square foot. Find the room dimensions that result in a minimum cost for the paint.

Step 1: Analysis



$V = xyz$

$668.25 = xyz$

$z = \frac{668.25}{xy}$

$SA = xy + 2z(x+y)$

Ceiling area xy

walls $2xz + 2yz$

Step 2: primary eq

$C(x,y,z) = .11xy + .06(2z(x+y))$

Step 3: Reduce primary

$C(x,y) = .11xy + \frac{.12(x+y) \cdot 668.25}{xy}$

$C(x,y) = .11xy + 80.19\left(\frac{1}{y} + \frac{1}{x}\right)$

Step 4: Optimize

$C_x(x,y) = .11y - \frac{80.19}{x^2} = 0$

$C_y(x,y) = .11x - \frac{80.19}{y^2} = 0$

In order for

$.11y - \frac{80.19}{x^2} = .11x - \frac{80.19}{y^2}$

x must equal y

So $.11x - \frac{80.19}{x^2} = 0$

$.11x = \frac{80.19}{x^2}$

$.11x^3 = 80.19$

$\sqrt[3]{x^3} = \sqrt[3]{729}$

$x = 9 = y$

$z = \frac{668.25}{9 \cdot 9}$

$z = 8.25$

Step 5: Conclusion

When the length and width are 9 feet and the height is 8.25 ft, the cost is minimized.

7. (20 POINTS) Consider the function $f(x,y) = -x^2 + 4xy - y^2 + 16x + 10$.

a. (10 POINTS) Find the critical points.

$f_x(x,y) = -2x + 4y + 16 = 0$

$f_y(x,y) = 4x - 2y = 0$

$-2x + 4y = -16$

$4x = 2y$

$4x = 2y$, $y = 2x$

$-2x + 4(2x) = -16$

$6x = -16$

$x = -8/3$, $y = 2(-8/3) = -\frac{16}{3}$

The critical point is $(-8/3, -16/3)$

$f(-8/3, -16/3) = -(-8/3)^2 + 4(-8/3)(-16/3) - (-16/3)^2 + 16(-8/3) + 10$

$= \frac{64}{9} + \frac{512}{9} - \frac{256}{9} - \frac{128}{3} + 10$

$= \frac{192}{9} - \frac{384}{9} + \frac{90}{9}$

$= -\frac{102}{9}$

$= -\frac{34}{3}$

b. (8 POINTS) Test for relative extrema and saddle points.

$f_{xx}(x,y) = -2 = f_{xx}(-8/3, -16/3)$

$f_{yy}(x,y) = -2 = f_{yy}(-8/3, -16/3)$

$f_{xy}(x,y) = 4 = f_{xy}(-8/3, -16/3)$

$d = (-2)(-2) - (4)^2$

$d = 4 - 16 = -12 < 0$

c. (2 POINTS) State your conclusion in words.

There is a saddle point at $(-8/3, -16/3, -34/3)$