DEFINITION OF UNIT TANGENT VECTOR

Let C be a smooth curve represented by $\mathbf{r}$ on an open interval I. The **unit tangent vector** $\mathbf{T}(t)$ at t is defined to be

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},\ \mathbf{r}'(t)\neq\mathbf{0}$$

The **tangent line to a curve** at a point is the line passing through point and parallel to the unit tangent vector.

Example 1: Find the unit tangent vector to the curve $\mathbf{r}(t)=e^t\cos t\mathbf{i}+e^t\mathbf{j}$ when $t=0$.

$\vec{r}'(t)=\left[e^t\cos t-e^t\sin t\right]\vec{i}+e^t\vec{j}$

$=e^t\left[(\cos t-\sin t)\vec{i}+\vec{j}\right]$

$\|\vec{r}'(t)\|=\sqrt{\left[e^t(\cos t-\sin t)\right]^2+\left(e^t\right)^2}$

$=\sqrt{e^{2t}\left[(\cos t-\sin t)^2+1\right]}$

$=e^t\sqrt{(\cos t-\sin t)^2+1}$

$\vec{T}'(t)=\dfrac{\cancel{e^t}\left[(\cos t-\sin t)\vec{i}+\vec{j}\right]}{\cancel{e^t}\sqrt{(\cos t-\sin t)^2+1}}$

$\vec{T}'(0)=\boxed{\dfrac{\vec{i}+\vec{j}}{\sqrt{2}}}$

Example 2: Consider the space curve $\mathbf{r}(t)=\left\langle t,t,\sqrt{4-t^2}\right\rangle$ at the point $\left(1,1,\sqrt{3}\right)$. a.

a. Find the unit tangent vector at the given point.

$\vec{T}'(t)=\dfrac{\left\langle 1,1,\frac{1}{\cancel{2}\sqrt{4-t^2}}(-\cancel{2}t)\right\rangle}{\sqrt{1^2+1^2+\left(-\frac{t}{\sqrt{4-t^2}}\right)^2}}=\dfrac{\left\langle 1,1,-\frac{t}{\sqrt{4-t^2}}\right\rangle}{\sqrt{2+\frac{t^2}{4-t^2}}}$

Find t: $x=t$, $x=1$ so $t=1$

$\vec{T}'(1)=\dfrac{\left\langle 1,1,-\frac{1}{\sqrt{3}}\right\rangle}{\sqrt{7/3}}=\boxed{\dfrac{\sqrt{21}}{3}\left\langle 1,1,-\frac{1}{\sqrt{3}}\right\rangle}$

b. Find a set of parametric equations for the line tangent to the space curve at the given point.

$a=1,\ b=1,\ c=-\frac{1}{\sqrt{3}}$

$x=at+x_0$ | $y=bt+y_0$ | $z=ct+z_0$

$x=t+1$ | $y=t+1$ | $z=-\frac{1}{\sqrt{3}}t+\sqrt{3}$

$$\boxed{\begin{aligned}x&=t+1\\ y&=t+1\\ z&=-\tfrac{1}{\sqrt{3}}t+\sqrt{3}\end{aligned}}$$

DEFINITION: PRINCIPAL UNIT NORMAL VECTOR

Let C be a smooth curve represented by $\mathbf{r}$ on an open interval I. If $\mathbf{T}'(t) \neq \mathbf{0}$, then the **principal unit normal vector** $\mathbf{N}(t)$ at t is defined to be

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

At any point on a curve, a unit normal vector is orthogonal to the unit tangent vector. The principal unit normal vector points in the direction in which the curve is turning.

Example 3: Find the principal unit normal vector to the curve $\mathbf{r}(t) = \ln t\mathbf{i} + (t+1)\mathbf{j}$ at the time $t = 2$.

$$\vec{r}'(t) = \frac{1}{t}\vec{i} + \vec{j}$$

$$\|r'(t)\| = \sqrt{\frac{1}{t^2} + 1} = \frac{\sqrt{1+t^2}}{t}$$

$$\vec{T}'(t) = \frac{t}{\sqrt{1+t^2}}\left\langle \frac{1}{t}, 1 \right\rangle = \left\langle \frac{1}{\sqrt{1+t^2}}, \frac{t}{\sqrt{1+t^2}} \right\rangle$$

$$\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{1+t^2}}\right) = -\frac{1}{\cancel{2}}\left(\frac{\cancel{2}t}{(1+t^2)^{3/2}}\right) = -\frac{t}{(1+t^2)^{3/2}}$$

$$\frac{\partial}{\partial t}\left(\frac{t}{\sqrt{1+t^2}}\right) = \frac{\sqrt{1+t^2} - t\left[\frac{1}{\cancel{2}}\left(\frac{\cancel{2}t}{\sqrt{1+t^2}}\right)\right]}{\left(\sqrt{1+t^2}\right)^2} = \frac{1+t^2-t^2}{(1+t^2)^{3/2}} = \frac{1}{(1+t^2)^{3/2}}$$

So $T'(t)$ is

$$\left\langle -\frac{t}{(1+t^2)^{3/2}}, \frac{1}{(1+t^2)^{3/2}} \right\rangle$$

$$\|\vec{T}'(t)\| = \sqrt{\left(\frac{-t}{(1+t^2)^{3/2}}\right)^2 + \left(\frac{1}{(1+t^2)^{3/2}}\right)^2} = \sqrt{\frac{t^2+1}{(1+t^2)^3}}$$

$$\vec{N}'(2) = \frac{\left\langle -\frac{5}{5^{3/2}}, \frac{1}{5^{3/2}} \right\rangle}{\frac{\sqrt{5}}{\sqrt{5^3}}} = \frac{\left\langle -\frac{5}{5^{3/2}}, \frac{1}{5^{3/2}} \right\rangle}{\frac{1}{5}} = \frac{5}{5^{3/2}}\langle -5, 1\rangle = \boxed{\frac{1}{\sqrt{5}}\langle -5, 1\rangle}$$

THEOREM: ACCELERATION VECTOR

If $\mathbf{r}(t)$ is the position vector for a smooth curve C and $\mathbf{N}(t)$ exists, then the acceleration vector $\mathbf{a}(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t)$ lies in the plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

THEOREM: TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

If $\mathbf{r}(t)$ is the position vector for a smooth curve C and $\mathbf{N}(t)$ exists, then the **tangential and normal components of acceleration are as follows**:

$$a_{\mathbf{T}} = D_t\left[\|\mathbf{v}\|\right] = \mathbf{a}\cdot\mathbf{T} = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|}$$

$$a_{\mathbf{N}} = \|\mathbf{v}\|\|\mathbf{T}'\| = \mathbf{a}\cdot\mathbf{N} = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^{\ 2}}$$

Note that $a_{\mathbf{N}} \ge 0$. The normal component of acceleration is also called the **centripetal component of acceleration.**

Example 4: Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$ for the plane curve $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k}$ at the time $t = 0$.

$\vec{r}'(t) = \langle e^t, -e^{-t}, 1\rangle = \vec{v}(t)$, $\|r'(t)\| = \sqrt{e^{2t}+e^{-2t}+1} = \sqrt{e^{2t}+\frac{1}{e^{2t}}+1}$

$= \sqrt{\dfrac{e^{4t}+e^{2t}+1}{e^{2t}}} = \dfrac{\sqrt{e^{4t}+e^{2t}+1}}{e^t}$

$\vec{r}''(t) = \langle e^t, e^{-t}, 0\rangle = \vec{a}(t)$

$\vec{T}(t) = \dfrac{e^t}{\sqrt{e^{4t}+e^{2t}+1}}\langle e^t, -e^{-t}, 1\rangle$

$\vec{T}(t) = \dfrac{\langle e^{2t}, -1, e^t\rangle}{\sqrt{e^{4t}+e^{2t}+1}}$

$\vec{N}(t) =$

$$\frac{\partial}{\partial t}\left(\frac{e^{2t}}{\sqrt{e^{4t}+e^{2t}+1}}\right) = \frac{2e^{2t}\sqrt{e^{4t}+e^{2t}+1} - e^{2t}\dfrac{(\cancel{4}^{2}e^{4t}+\cancel{2}e^{2t})}{\cancel{2}\sqrt{e^{4t}+e^{2t}+1}}}{e^{4t}+e^{2t}+1}$$

$$= \frac{2e^{2t}(e^{4t}+e^{2t}+1) - e^{2t}(2e^{4t}+e^{2t})}{(e^{4t}+e^{2t}+1)^{3/2}}$$

$$= \frac{e^{2t}\left[\cancel{2e^{4t}}+\cancel{2e^{2t}}+2-(\cancel{2e^{4t}}+\cancel{2e^{2t}})\right]}{(e^{4t}+e^{2t}+1)^{3/2}}$$

$$= \frac{2e^{2t}}{(e^{4t}+e^{2t}+1)^{3/2}}$$

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$$\frac{\partial}{\partial t}\left(-\left(e^{4t}+e^{2t}+1\right)^{-1/2}\right) = +\frac{1}{\cancel{2}}\left(e^{4t}+e^{2t}+1\right)^{-3/2}\left(\cancel{4}^{\,2}e^{4t}+\cancel{2}e^{2t}\right)$$

$$= \frac{2e^{4t}+e^{2t}}{\left(e^{4t}+e^{2t}+1\right)^{3/2}}$$

$$\frac{\partial}{\partial t}\left(\frac{e^{t}}{\sqrt{e^{4t}+e^{2t}+1}}\right) = \frac{e^{t}\sqrt{e^{4t}+e^{2t}+1} - \dfrac{e^{t}\left(\cancel{4}^{\,2}e^{4t}+\cancel{2}e^{2t}\right)}{\cancel{2}\sqrt{e^{4t}+e^{2t}+1}}}{e^{4t}+e^{2t}+1}$$

$$= \frac{e^{t}\left(e^{4t}+e^{2t}+1\right)-e^{t}\left(e^{4t}+e^{2t}\right)}{\left(e^{4t}+e^{2t}+1\right)^{3/2}}$$

$$= \frac{e^{t}\left[\left(\cancel{e^{4t}}+\cancel{e^{2t}}+1\right)-\left(\cancel{e^{4t}}+\cancel{e^{2t}}\right)\right]}{\left(e^{4t}+e^{2t}+1\right)^{3/2}}$$

$$= \frac{e^{t}}{\left(e^{4t}+e^{2t}+1\right)^{3/2}}$$

So $\vec{T}'(t) = \left\langle \dfrac{2e^{2t}}{\left(e^{4t}+e^{2t}+1\right)^{3/2}}, \dfrac{2e^{4t}+e^{2t}}{\left(e^{4t}+e^{2t}+1\right)^{3/2}}, \dfrac{e^{t}}{\left(e^{4t}+e^{2t}+1\right)^{3/2}} \right\rangle$

$$\|\vec{T}'(t)\| = \sqrt{\frac{4e^{4t}}{\left(e^{4t}+e^{2t}+1\right)^{3}} + \frac{4e^{8t}+4e^{6t}+e^{4t}}{\left(e^{4t}+e^{2t}+1\right)^{3}} + \frac{e^{2t}}{\left(e^{4t}+e^{2t}+1\right)^{3}}}$$

$$= \sqrt{\frac{4e^{8t}+4e^{6t}+5e^{4t}+e^{2t}}{\left(e^{4t}+e^{2t}+1\right)^{3}}}$$

So...

$$\vec{N}(t) = \frac{\cancel{(e^{4t}+e^{2t}+1)^{3/2}}}{(4e^{8t}+4e^{6t}+5e^{4t}+e^{2t})^{1/2}} \left\langle \frac{2e^{2t}}{\cancel{(e^{4t}+e^{2t}+1)^{3/2}}}, \frac{2e^{4t}+e^{2t}}{\cancel{(e^{4t}+e^{2t}+1)^{3/2}}}, \frac{e^{t}}{\cancel{(e^{4t}+e^{2t}+1)^{3/2}}} \right\rangle$$

$$= \boxed{\frac{1}{(4e^{8t}+4e^{6t}+5e^{4t}+e^{2t})^{1/2}}\left(2e^{2t}\vec{i} + (2e^{4t}+e^{2t})\vec{j} + e^{t}\vec{k}\right)}$$

$$a_{\vec{T}} = \frac{\vec{v}\cdot\vec{a}}{\|\vec{v}\|}$$

$$= \frac{(e^{t})(e^{t}) + (-e^{-t})(e^{-t}) + (1)(0)}{\sqrt{e^{2t}+e^{-2t}+1}}$$

$$= \frac{e^{2t}-e^{-2t}}{\sqrt{e^{2t}+e^{-2t}+1}}$$

$$= \frac{e^{2t}-\frac{1}{e^{2t}}}{\sqrt{e^{2t}+\frac{1}{e^{2t}}+1}}$$

$$= \frac{\frac{e^{4t}+1}{e^{2t}}}{\frac{\sqrt{e^{4t}+e^{2t}+1}}{e^{t}}}$$

$$= \boxed{\frac{e^{4t}+1}{e^{t}\sqrt{e^{4t}+e^{2t}+1}}}$$

$$a_{\vec{N}} = \sqrt{\|\vec{a}\|^2 - a_T^2}$$

$$= \sqrt{\left(\sqrt{e^{2t}+e^{-2t}}\right)^2 - \left(\frac{e^{4t}+1}{e^{t}\sqrt{e^{4t}+e^{2t}+1}}\right)^2}$$

$$= \sqrt{\frac{e^{4t}+1}{e^{2t}} - \frac{(e^{8t}+2e^{4t}+1)}{e^{2t}(e^{4t}+e^{2t}+1)}}$$

$$= e^{8t}+e^{6t}+e^{4t}+e^{4t}+e^{2t}$$