Today	Monday	Wednesday	Next Friday
Lecture 15.5	Review	Exam 5 - 15.1-15.5	Review

Final Exam

Monday, 5/16
1-3

Alternate times

Monday, 10:30-12:30
room 563 B

Tuesday, 10:30-12:30
room 343

Thursday, 8-10
room 1620

you must e-mail me by next Wednesday

* **Extra Credit due next Friday, 5/13**

- Only turn in fully attempted chapter reviews

When you are done with your homework you should be able to…

- π Understand the definition of and sketch a parametric surface
- π Find a set of parametric equations to represent a surface
- π Find a normal vector and a tangent plane to a parametric surface
- π Find the area of a parametric surface

both positive $\Rightarrow$ QI

$2\sin t = 1$ and $2\cos t = \sqrt{3}$

$t = \pi/6$

Warm-up:

1. Find the unit tangent vector $\mathbf{T}(t)$ and find a set of parametric equations for the line tangent to the space curve $\mathbf{r}(t) = \langle 2\sin t, 2\cos t, 4\sin^2 t \rangle$ at the point $(1, \sqrt{3}, 1)$.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

$$\vec{T}(t) = \frac{\langle 2\cos t, -2\sin t, 8\sin t\cos t \rangle}{\sqrt{4\cos^2 t + 4\sin^2 t + 64\sin^2 t\cos^2 t}}$$

$$\vec{T}(t) = \frac{\cancel{2}\langle \cos t, -\sin t, 4\sin t\cos t \rangle}{\cancel{2}\sqrt{1 + 16\sin^2 t\cos^2 t}}$$

$$\vec{T}(\pi/6) = \frac{\langle \cos \pi/6, -\sin \pi/6, 4\sin\pi/6 \cos\pi/6 \rangle}{\sqrt{1 + 16(\sin\pi/6)^2(\cos\pi/6)^2}}$$

$$\vec{T}(\pi/6) = \frac{\langle \sqrt{3}/2, -1/2, 4(1/2)(\sqrt{3}/2) \rangle}{\sqrt{1 + 16(\frac{1}{4})(\frac{3}{4})}}$$

$$\vec{T}(\pi/6) = \frac{\frac{1}{2}\langle \sqrt{3}, -1, 2\sqrt{3} \rangle}{2}$$

$$\vec{T}(\pi/6) = \frac{1}{4}\langle \underset{a}{\sqrt{3}}, \underset{b}{-1}, \underset{c}{2\sqrt{3}} \rangle$$

$$\boxed{\begin{aligned} x &= \sqrt{3}t + 1 \\ y &= -t + \sqrt{3} \\ z &= 2\sqrt{3}t + 1 \end{aligned}}$$

How do you represent a curve in the plane by a vector-valued function?

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

How do you represent a curve in space by a vector-valued function?

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

DEFINITION OF PARAMETRIC SURFACE

Let x, y and z be functions of u and v that are continuous on a domain D in the uv-plane. The set of points (x, y, z) given by

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \qquad \text{Parametric surface}$$

is called a **parametric surface**. The equations

$$x = x(u,v), \quad y = y(u,v), \quad \text{and} \quad z = z(u,v) \qquad \text{Parametric equations}$$

are the **parametric equations** for the surface.

Example 1: Find the rectangular equation for the surface by eliminating the parameters from the vector-valued function. Identify the surface and sketch its graph.

$$\mathbf{r}(u,v) = 2u\cos v\mathbf{i} + 2u\sin v\mathbf{j} + \frac{1}{2}u^2\mathbf{k}$$

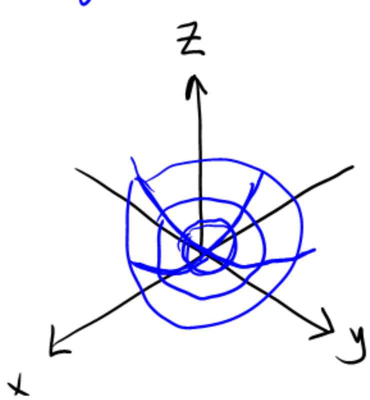
$x = x(u,v)$ $\quad$ $y = y(u,v)$ $\quad$ $z = z(u,v)$

$x = 2u\cos v$ $\quad$ $y = 2u\sin v$ $\quad$ $z = \frac{1}{2}u^2$

$x^2 = 4u^2\cos^2 v$ $\quad$ $y^2 = 4u^2\sin^2 v$

$x^2 + y^2 = 4u^2\cos^2 v + 4u^2\sin^2 v$

$x^2 + y^2 = 4u^2$ so, $x^2 + y^2 = 4u^2$ and $8z = 4u^2$



$8z = x^2 + y^2$

$z = \frac{1}{8}(x^2 + y^2)$

paraboloid

Example 2: Find a vector-valued function whose graph is the indicated surface.

The plane $x + y + z = 6$

$z = 6 - x - y$

Let $x = u$ so $z = 6 - u - v$

Let $y = v$ $\vec{r}(u,v) = u\hat{i} + v\hat{j} + (6-u-v)\hat{k}$

Example 3: Write a set of parametric equations for the surface of revolution obtained by revolving the graph of $y = x^{3/2}$, $0 \le x \le 4$ about the *x*-axis.

$$\boxed{\left.\begin{aligned} x &= u \\ y &= u^{3/2}\cos v \\ z &= u^{3/2}\sin v \end{aligned}\right\} \begin{gathered} 0 \le u \le 4 \\ \text{and} \\ 0 \le v \le 2\pi \end{gathered}}$$

Let $x = u$

$y = f(u)\cos v$

$z = f(u)\sin v$

for revolving about the x-axis

Example 4: Find an equation of the tangent plane to the surface represented by the vector-valued function $\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + \sqrt{uv}\,\mathbf{k}$ at the point $(1,1,1)$.

$\vec{r}_u = \hat{i} + \frac{v}{2\sqrt{uv}}\hat{k}$

$\vec{r}_v = \hat{j} + \frac{u}{2\sqrt{uv}}\hat{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{v}{2\sqrt{uv}} \\ 0 & 1 & \frac{u}{2\sqrt{uv}} \end{vmatrix}$$

$$= -\begin{vmatrix} \hat{j} & \hat{k} \\ 1 & \frac{u}{2\sqrt{uv}} \end{vmatrix} 1 - \begin{vmatrix} \hat{i} & \hat{j} \\ 0 & 1 \end{vmatrix} \frac{v}{2\sqrt{uv}}$$

$$= -\left(\frac{u}{2\sqrt{uv}}\right)\hat{j} + \hat{k} - \frac{v}{2\sqrt{uv}}\hat{i}$$

$$= -\frac{v}{2\sqrt{uv}}\hat{i} - \frac{u}{2\sqrt{uv}}\hat{j} + \hat{k}$$

at: $(1,1,1)$

$$\vec{r}_u \times \vec{r}_v = -\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} + k$$

$$= -\frac{1}{2}\langle 1, 1, -2 \rangle$$

$a = 1$

$b = 1$

$c = -2$

$1(x-1) + 1(y-1) - 2(z-1) = 0$

$x - 1 + y - 1 - 2z + 2 = 0$

$$\boxed{x + y - 2z = 0}$$

AREA OF A PARAMETRIC SURFACE

Let S be a smooth parametric surface $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$ defined over an open region D in the *uv*-plane. If each point on the surface S corresponds to exactly one point in the domain D, then the **surface area** of S is given by

$$\text{Surface Area} = \int_S\int dS = \int_D\int \left\|\mathbf{r}_u \times \mathbf{r}_v\right\| dA$$

$$\text{where } \mathbf{r}_u = \frac{dx}{du}\mathbf{i} + \frac{dy}{du}\mathbf{j} + \frac{dz}{du}\mathbf{k} \text{ and } \mathbf{r}_v = \frac{dx}{dv}\mathbf{i} + \frac{dy}{dv}\mathbf{j} + \frac{dz}{dv}\mathbf{k}$$

Example 5: Find the area of the surface over the part of the paraboloid $\mathbf{r}(u,v) = 4u\cos v\mathbf{i} + 4u\sin v\mathbf{j} + u^2\mathbf{k}$, where $0 \le u \le 2$ and $0 \le v \le 2\pi$.

$\vec{r}_u = 4\cos v\,\hat{i} + 4\sin v\,\hat{j} + 2u\hat{k}$

$\vec{r}_v = -4u\sin v\,\hat{i} + 4u\cos v\,\hat{j}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4\cos v & 4\sin v & 2u \\ -4u\sin v & 4u\cos v & 0 \end{vmatrix}$$

$$= (0 - 8u^2\cos v)\hat{i} - (0 + 8u^2\sin v)\hat{j} + (16u\cos^2 v + 16u\sin^2 v)\hat{k}$$

$$= -8u^2\cos v\,\hat{i} - 8u^2\sin v\,\hat{j} + 16u\,\hat{k}$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{64u^4\cos^2 v + 64u^4\sin^2 v + 256u^2}$$

$$= \sqrt{64u^4 + 256u^2}$$

$$= \sqrt{64u^2(u^2+4)}$$

$$= 8u\sqrt{u^2+4}$$

$$SA = \int_0^{2\pi}\int_0^2 8u\sqrt{u^2+4}\,du\,dv$$

$$= 4\int_0^{2\pi} \frac{2}{3}(u^2+4)^{3/2}\Big|_0^2$$

$$= \frac{8}{3}\int_0^{2\pi}(8\sqrt{8} - 8)\,dv$$

$$= \frac{8}{3}\,8(2\sqrt{2} - 1)v\Big|_0^{2\pi}$$

$$= \frac{64}{3}(2\sqrt{2}-1)(2\pi - 0)$$

$$= \boxed{\frac{128\pi}{3}(2\sqrt{2}-1)}$$