When you are done with your homework you should be able to…

- π Use Green's Theorem to evaluate a line integral
- π Use alternative forms of Green's Theorem

Warm-up:

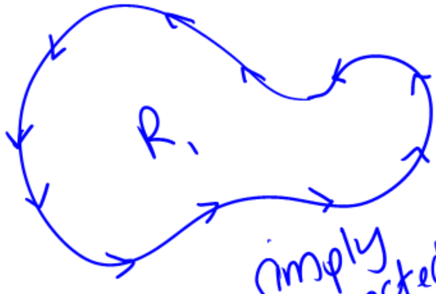
1. Represent the plane curve $2x-3y+5=0$ by a vector-valued function.

2. Determine whether the vector field $\mathbf{F}$ is conservative. If it is, find a potential function for the vector field.

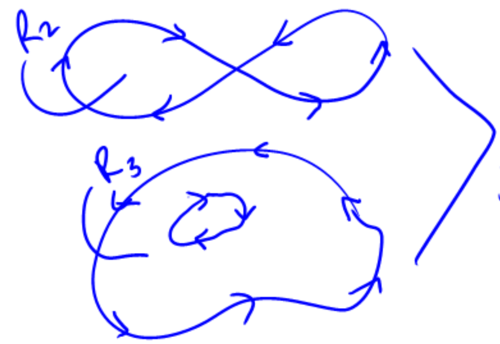
$$\mathbf{F}(x,y,z)=\frac{x}{x^2+y^2}\mathbf{i}+\frac{y}{x^2+y^2}\mathbf{j}+\mathbf{k}$$

SIMPLE CURVES:

- A curve *C* given by $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j},\quad a\le t\le b$ is **simple** if it does not cross itself—that is $\mathbf{r}(c)\neq\mathbf{r}(d)$ for all *c* and *d* in the open interval (a,b)
- A plane region *R* is **simply connected** if its boundary consists of one simple closed curve



simply connected



not simply connected

GREEN'S THEOREM

Let R be a simply connected region with a piecewise smooth boundary C, oriented counterclockwise (that is, C is traversed once so that the region R always lies to the left). If M and N have continuous partial derivatives in an open region containing R, then

$$\int_C M\,dx+N\,dy=\int_R\int\left(\frac{dN}{dx}-\frac{dM}{dy}\right)dA$$

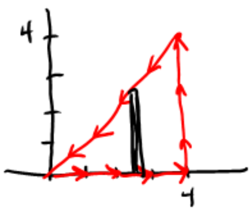
Example 1: Verify Green's Theorem by evaluating both integrals

$\int_C y^2dx+x^2dy=\int_R\int\left(\frac{dN}{dx}-\frac{dM}{dy}\right)dA$ for the given path.

New way: Green's Thm

$M=y^2 \qquad N=x^2$

Old Way



C: triangle with vertices $(0,0)$, $(4,0)$, $(4,4)$

$\frac{dM}{dy}=2y$ and $\frac{dN}{dx}=2x$

$\int_C y^2dx+x^2dy=\int_0^4\int_0^x(2x-2y)\,dy\,dx$

$=\int_0^4(2xy-y^2)\Big|_0^x dx=\int_0^4 x^2dx=\frac{x^3}{3}\Big|_0^4$

$=\boxed{64/3}$

$x(t)=t,\ dx=dt$
$y(t)=0,\ dy=0$

$x(t)=4,\ dx=0$
$y(t)=t-4,\ dy=dt$

$x(t)=12-t,\ dx=-dt$
$y(t)=12-t,\ dy=-dt$

$$\vec{r}(t)=\begin{cases} t\,\hat{i}, & 0\le t\le 4\\ 4\hat{i}+(t-4)\hat{j}, & 4\le t\le 8\\ (12-t)\hat{i}+(12-t)\hat{j}, & 8\le t\le 12\end{cases}$$

$\int_C y^2dx+x^2dy=\int_0^4(0)^2dt+t^2\cdot 0+\int_4^8(t-4)^2\cdot 0+4^2dt+\underbrace{\int_8^{12}(12-t)^2(-dt)+(12-t^2)(-dt)}_{+2\int_8^{12}(12-t)^2dt}$

$=0+16t\Big|_4^8+\frac{2(12-t)^3}{3}\Big|_8^{12}=64-\frac{128}{3}=\boxed{\frac{64}{3}}$

Example 2: Use Green's Theorem to evaluate the integral $\int_C (y-x)\,dx + (2x-y)\,dy$ for the given path.

$A_{\text{ellipse}} = \pi ab$

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

OR

$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$

$M = y - x \qquad N = 2x - y$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 2$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2 - 1 = 1$

$C:\ x = 2\cos\theta,\ y = \sin\theta$

$\frac{x}{2} = \cos\theta$

so $\cos^2\theta + \sin^2\theta = 1$

$\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$

$a = 2,\ b = 1$

$A = \pi ab = \pi(2)(1) = 2\pi$

$\int_C (y-x)\,dx + (2x-y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$= 1(2\pi)$

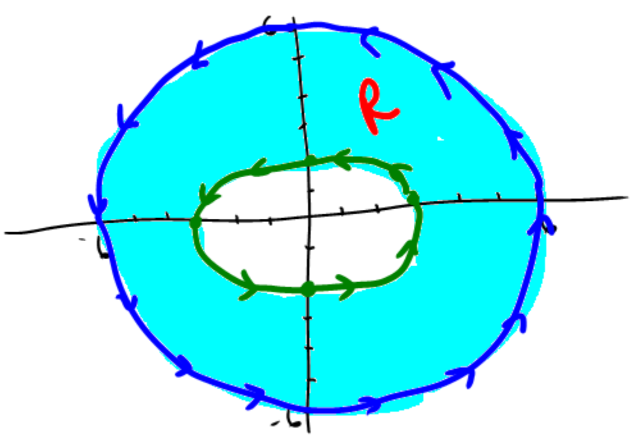
$= \boxed{2\pi}$

Example 3: Use Green's Theorem to evaluate the line integral.

$C = C_1 + C_2$

$\int_C \left(e^{-x^2/2} - y\right)dx + \left(e^{-y^2/2} + x\right)dy$

C: boundary of the region lying between the graphs of the circle $x = 6\cos\theta,\ y = 6\sin\theta$ and the ellipse $x = 3\cos\theta,\ y = 2\sin\theta$



$M = e^{-x^2/2} - y \qquad N = e^{-y^2/2} + x$

$\frac{\partial M}{\partial y} = -1 \qquad \frac{\partial N}{\partial x} = 1$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - (-1) = 2$

Area of R is the area of the circle minus the area of the ellipse

$A = \pi \cdot 6^2 - \pi(3)(2) = 30\pi$

$\int_C \left(e^{-x^2/2} - y\right)dx + \left(e^{-y^2/2} + x\right)dy$

$= \iint_R (2)\,dA$

$= 30\pi(2) = \boxed{60\pi}$

THEOREM: LINE INTEGRAL FOR AREA

If R is a plane region bounded by a piecewise smooth simple closed curve C, oriented counterclockwise, then the area of R is given by

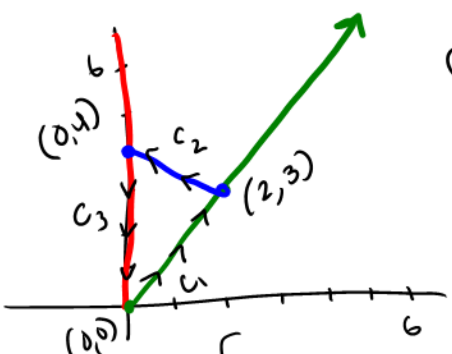
$$A = \frac{1}{2}\int_C x\,dy - y\,dx$$

Example 4: Use a line integral to find the area of the region R.

$y = -\frac{1}{2}x + 4$

R: triangle bounded by the graphs of $x = 0$, $3x - 2y = 0$, $x + 2y = 8$

$y = \frac{3}{2}x$



$C = C_1 + C_2 + C_3$

C_1: $y = \frac{3}{2}x$, $dy = \frac{3}{2}dx$, $0 \le x \le 2$

C_2: $y = -\frac{1}{2}x + 4$, $dy = -\frac{1}{2}dx$, $2 \le x \le 0$

C_3: $x = 0$, $dx = 0$, $0 \le x \le 0$

$$A = \frac{1}{2}\left[\int_0^2 x\left(\frac{3}{2}dx\right) - \left(\frac{3}{2}x\right)dx + \int_2^0 \overbrace{x\left(-\frac{1}{2}dx\right)}^{-\frac{x}{2}dx} - \overbrace{\left(-\frac{1}{2}x + 4\right)dx}^{+\frac{x}{2}dx - 4dx} + \int_0^0 0\,dy - y \cdot 0\right]$$

$$A = \frac{1}{2}\left[\int_0^2 0\,dx + \int_2^0 (-4)\,dx + 0\right]$$

$$A = \frac{1}{2}\left(-4x\Big|_2^0\right)$$

$$A = \frac{1}{2}\left(-4 \cdot 0 - (-4) \cdot 2\right)$$

$$A = \frac{1}{2} \cdot 8$$

$$\boxed{A = 4 \text{ sq. units}}$$

ALTERNATIVE FORMS OF GREEN'S THEOREM

If $\mathbf{F}$ is a vector field in the plane, you can write $\mathbf{F}(x,y,z)=M\mathbf{i}+N\mathbf{j}+0\mathbf{k}$. Thus, the

$$\text{curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ M & N & 0 \end{vmatrix}=-\frac{dN}{dz}\mathbf{i}+\frac{dM}{dz}\mathbf{j}+\left(\frac{dN}{dx}-\frac{dM}{dy}\right)\mathbf{k}$$

and $(\text{curl } \mathbf{F})\cdot\mathbf{k}=\frac{dN}{dx}-\frac{dM}{dy}$. With appropriate conditions on $\mathbf{F}$, C and R, you can write Green's Theorem in the vector form

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_R\int\left(\frac{dN}{dx}-\frac{dM}{dy}\right)dA$$
$$=\int_R\int(\text{curl } \mathbf{F})\cdot\mathbf{k}\,dA$$

(First alternative form)

Assume the same conditions for $\mathbf{F}$, C and R. Using the arc length parameter s for *C*, you have $\mathbf{r}(s)=x(s)\mathbf{i}+y(s)\mathbf{j}$. So a unit tangent vector $\mathbf{T}$ to the curve *C* is given by $\mathbf{r}'(s)=x'(s)\mathbf{i}+y'(s)\mathbf{j}$ and the outward unit normal vector $\mathbf{N}$ can be written as $\mathbf{N}=y'(s)\mathbf{i}-x'(s)\mathbf{j}$. So for $\mathbf{F}(x,y)=M\mathbf{i}+N\mathbf{j}$ we have,

$$\int_C \mathbf{F}\cdot\mathbf{N}\,ds=\int_a^b(M\mathbf{i}+N\mathbf{j})\cdot\left(y'(s)\mathbf{i}-x'(s)\mathbf{j}\right)ds$$
$$=\int_a^b\left(M\frac{dy}{ds}-N\frac{dx}{ds}\right)ds$$
$$=\int_C Mdy-Ndx$$
$$=\int_C -Ndx+Mdy$$

(Second alternative form)

$$=\int_R\int\left(\frac{dM}{dx}+\frac{dN}{dy}\right)dA$$
$$=\int_R\int \text{div } \mathbf{F}\,dA$$