5/2/11	Wednesday	Friday	Exam on Monday 5/9/11
• 15.3	15.4	15.5	Next Wed, Fri → review for Final

When you are done with your homework you should be able to…

- π Understand and use the Fundamental Theorem of Line Integrals
- π Understand the concept of independence of path
- π Understand the concept of conservation of energy

Warm-up: Show that the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ is the same for each parametric representation of C.

$$\mathbf{F}(x,y)=\left(x^2+y^2\right)\mathbf{i}-x\mathbf{j}$$

(a) $\mathbf{r}_1(t)=t\mathbf{i}+\sqrt{t}\mathbf{j},\ 0\le t\le 4$

(b) $\mathbf{r}_2(w)=w^2\mathbf{i}+w\mathbf{j},\ 0\le w\le 2$

a) $x(t)=t,\ x'(t)=1$

$y(t)=\sqrt{t},\ y'(t)=\frac{1}{2\sqrt{t}}$

$\vec{F}(x(t),y(t))=(t^2+t)\hat{i}-t\hat{j}$

$\vec{r}'(t)=\hat{i}+\frac{1}{2\sqrt{t}}\hat{j}$

$\vec{F}(x(t),y(t))\cdot\vec{r}'(t)=\left\langle (t^2+t),-t\right\rangle\cdot\left\langle 1,\frac{1}{2\sqrt{t}}\right\rangle$

$=(t^2+t)(1)+(-t)\left(\frac{1}{2\sqrt{t}}\right)$

$=t^2+t-\frac{1}{2}\sqrt{t}$

$\int_0^4\left(t^2+t-\frac{1}{2}t^{1/2}\right)dt=\frac{t^3}{3}+\frac{t^2}{2}-\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}t^{3/2}\Big|_0^4$

$=\frac{64}{3}+8-\frac{1}{3}\cdot 8=\boxed{\frac{80}{3}}$

b) $x(w)=w^2,\ x'(w)=2w$

$y(w)=w,\ y'(w)=1$

$\vec{F}(x(w),y(w))=(w^4+w^2)\hat{i}-w^2\hat{j}$

$\vec{r}'(w)=2w\hat{i}+\hat{j}$

$\vec{F}(x(w),y(w))\cdot\vec{r}'(w)=2w^5+2w^3-w^2$

$\int_0^2(2w^5+2w^3-w^2)\,dw=\left(\frac{w^6}{3}+\frac{w^4}{2}-\frac{w^3}{3}\right)\Big|_0^2$

$=\frac{64}{3}+8-\frac{8}{3}$

$=\boxed{\frac{80}{3}}$

FUNDAMENTAL THEOREM OF LINE INTEGRALS

Let C be a piecewise smooth curve lying in an open region R and given by $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j},\ a\le t\le b$. If $\mathbf{F}(x,y)=M\mathbf{i}+N\mathbf{j}$ is conservative in R, and M and N are continuous in R, then

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C \nabla f\cdot d\mathbf{r}=f\left(x(b),y(b)\right)-f\left(x(a),y(a)\right)$$

where f is a potential function of $\mathbf{F}$. That is, $\mathbf{F}(x,y)=\nabla f(x,y)$.

extends to space as well

Example 1: Find the value of the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$.

$M=1,\ N=z,\ P=y$

$\mathbf{F}(x,y,z)=\mathbf{i}+z\mathbf{j}+y\mathbf{k}$

(a) $\mathbf{r}_1(t)=\cos t\mathbf{i}+\sin t\mathbf{j}+t^2\mathbf{k},\ 0\le t\le \pi$

(b) $\mathbf{r}_2(t)=(1-2t)\mathbf{i}+\pi^2 t\mathbf{k},\ 0\le t\le 1$

$\frac{\partial P}{\partial y}=1$, $\frac{\partial N}{\partial z}=1$ $\Rightarrow$ $\frac{\partial P}{\partial y}=\frac{\partial N}{\partial z}$ ✓

$\frac{\partial P}{\partial x}=0$, $\frac{\partial M}{\partial z}=0$ $\Rightarrow$ $\frac{\partial P}{\partial x}=\frac{\partial M}{\partial z}$ ✓

$\frac{\partial N}{\partial x}=0$, $\frac{\partial M}{\partial y}=0$ $\Rightarrow$ $\frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}$ ✓

So $\vec{F}$ is conservative

So we can use the Fundamental Theorem of Line Integrals

↳ we need the potential function

Potential Function

$\int M\,dx=\int 1\,dx=x+g(y,z)$

$\int N\,dy=\int z\,dy=yz+h(x,z)$

$\int P\,dz=\int y\,dz=yz+k(x,y)$

$g(y,z)=h(x,z)=yz$

$k(x,y)=K$

$f(x,y,z)=x+yz+K$

$\int \vec{F}\cdot d\vec{r}=f(x(b),y(b),z(b))-f(x(a),y(a),z(a))$

a) $x(t)=\cos t,\ y(t)=\sin t,\ z(t)=t^2,\ a=0,\ b=\pi$

$\int \vec{F}\cdot d\vec{r}=f(\cos\pi,\sin\pi,\pi^2)-f(\cos 0,\sin 0,0^2)$

$=f(-1,0,\pi^2)-f(1,0,0)$

$=\left(x+yz+K\right)\Big|_{(1,0,0)}^{(-1,0,\pi^2)}$

$=(-1+0\cdot\pi^2+K)-(1+0\cdot 0+K)$

$=\boxed{-2}$

b) $x(t)=1-2t$, $y(t)=0$, $z(t)=\pi^2 t$, $a=0,\ b=1$

$x(1)=-1$, $y(1)=0$, $z(1)=\pi^2$

$x(0)=1$, $y(0)=0$, $z(0)=0$

$\int \vec{F}\cdot d\vec{r}=\left(x+yz+K\right)\Big|_{(1,0,0)}^{(-1,0,\pi^2)}$

$=\boxed{-2}$

THEOREM: INDEPENDENCE OF PATH AND CONSERVATIVE VECTOR FIELDS

If $\mathbf{F}$ is continuous on an open connected region, then the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path if and only if $\mathbf{F}$ is conservative.

THEOREM: EQUIVALENT CONDITIONS

Let $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ have continuous first partial derivatives in an open connected region R, and let C be a piecewise smooth curve in R. The following conditions are equivalent:

1. $\mathbf{F}$ is conservative. That is, $\mathbf{F} = \nabla f$ for some function f.
2. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.
3. $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every **closed** curve C in R.

Example 2: Evaluate the line integral using the Fundamental Theorem of Line Integrals.

$$\int_C \left[2(x+y)\mathbf{i} + 2(x+y)\mathbf{j}\right] \cdot d\mathbf{r}$$

C: smooth curve from $(-2,2)$ to $(4,3)$

$\frac{\partial M}{\partial y} = 2 = \frac{\partial N}{\partial x}$

so $\vec{F}$ is Conservative

$M = N = 2(x+y)$

$\int M\,\partial x = 2\int (x+y)\,\partial x$

$= 2\left(\frac{x^2}{2} + xy\right) + g(y) = x^2 + 2xy$

$\int N\,dy = 2\left(xy + \frac{y^2}{2}\right) + g(x) = 2xy + y^2$

$f(x,y) = x^2 + 2xy + y^2 = (x+y)^2 \Big|_{(-2,2)}^{(4,3)}$

$= (7)^2 - (0)^2$

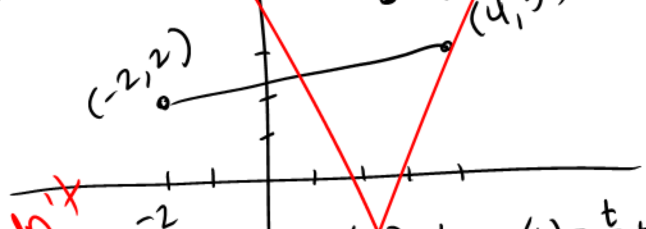
$= 49.$

$x(t) = 2(x+y)$

$m = \frac{1}{6}$

$y - 3 = \frac{1}{6}(x-4)$

$y = \frac{x}{6} + \frac{7}{3}$



$x(t) = t$, $y(t) = \frac{t}{6} + \frac{7}{3}$

$\vec{r}(t) = t\,\hat{i} + \left(\frac{t}{6} + \frac{7}{3}\right)\hat{j}$

$-2 \le t \le 4$

$x(4) = 4$, $x(-2) = -2$

$y(4) = 3$, $y(-2) = 2$

we didn't need this

Example 4: Evaluate the line integral using the Fundamental Theorem of Line Integrals.

$M = \frac{2x}{(x^2+y^2)^2}$ $N = \frac{2y}{(x^2+y^2)^2}$

$$\int_C \frac{2x}{\left(x^2+y^2\right)^2}dx + \frac{2y}{\left(x^2+y^2\right)^2}dy$$

C: circle $\left(x-4\right)^2+\left(y-5\right)^2=9$

clockwise from $(7,5)$ to $(1,5)$

$$\frac{\partial N}{\partial x} \overset{?}{=} \frac{\partial M}{\partial y}$$

$$\frac{2y\cdot(-2)(2x)}{(x^2+y^2)^3} \overset{?}{=} \frac{2x\cdot(-2)(2y)}{(x^2+y^2)^3}$$

yes so we can use the fund. thm. of Line integrals

$$\int_C M\,dx + N\,dy = -\frac{1}{x^2+y^2}\Bigg|_{(7,5)}^{(1,5)}$$

$$= -\frac{1}{(1)^2+(5)^2} - \left(-\frac{1}{(7)^2+(5)^2}\right)$$

$$= -\frac{1}{26} + \frac{1}{74}$$

$$= \boxed{-\frac{12}{481}}$$

Potential function

$$\int M\,dx = \int \frac{2x}{(x^2+y^2)^2}dx = -\frac{1}{x^2+y^2} + g(y)$$

$$\int N\,dy = \int \frac{2y}{(x^2+y^2)^2}dy = -\frac{1}{x^2+y^2} + h(x)$$

$$f(x,y) = \underbrace{-\frac{1}{x^2+y^2}}_{\text{only need this part}} + K$$

only need this part since constant will zero out