4/29/11

- Warm-up using 15.2 W.S.
- Finish 15.1
- Lecture 15.2

Monday	Next Friday
15.3	15.5
Wednesday	
15.4	

When you are done with your homework you should be able to…

- π Understand and use the concept of a piecewise smooth curve
- π Write and evaluate a line integral
- π Write and evaluate a line integral of a vector field
- π Write and evaluate a line integral in differential form

Warm-up:

1. Represent the plane curve $2x-3y+5=0$ by a vector-valued function.

$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

$y = \frac{2}{3}x + \frac{5}{3}$

let $x = t \rightarrow y = \frac{2}{3}t + \frac{5}{3}$

$x(t) = t \quad y(t) = \frac{2}{3}t + \frac{5}{3}$

$\boxed{\vec{r}(t) = t\hat{i} + \left(\frac{2}{3}t + \frac{5}{3}\right)\hat{j}}$

2. Determine whether the vector field $\mathbf{F}$ is conservative. If it is, find a potential function for the vector field.

$$\mathbf{F}(x,y,z) = \frac{x}{x^2+y^2}\mathbf{i} + \frac{y}{x^2+y^2}\mathbf{j} + \mathbf{k}$$

$M = \frac{x}{x^2+y^2}, N = \frac{y}{x^2+y^2}, P = 1$

$\frac{\partial P}{\partial y} = 0, \quad \frac{\partial N}{\partial z} = 0 \quad \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$ ✓

$\frac{\partial P}{\partial x} = 0, \quad \frac{\partial M}{\partial z} = 0 \quad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}$ ✓

$\frac{\partial N}{\partial x} = -\frac{2xy}{(x^2+y^2)^2}, \quad \frac{\partial M}{\partial y} = -\frac{2yx}{(x^2+y^2)^2} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ ✓

$\boxed{\text{So } \vec{F} \text{ is conservative}}$

$\int M\,dx = \frac{1}{2}\int \frac{2x}{x^2+y^2}\,dx = \frac{1}{2}\ln(x^2+y^2) + g(y,z)$

$\int N\,dy = \frac{1}{2}\int \frac{2y}{x^2+y^2}\,dy = \frac{1}{2}\ln(x^2+y^2) + h(x,z)$

$\int P\,dz = \int 1\,dz = z + k(x,y)$

$g(y,z) = h(x,z) = z$

$k(x,y) = K$

Potential function:

$\boxed{f(x,y,z) = \frac{1}{2}\ln(x^2+y^2) + z + K}$

PIECEWISE SMOOTH CURVES:

- The work done by gravity on an object moving between two points in the field is independent of the path taken by the object
  - One constraint is that the **path** must be a piecewise smooth curve
- Recall that a plane curve *C* given by $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j},\quad a\le t\le b$ is smooth if $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous on $[a,b]$ and not simultaneously 0 on (a,b).
  Similarly, a space curve *C* given by $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k},\quad a\le t\le b$ is smooth if $\frac{dx}{dt}, \frac{dy}{dt}$ and $\frac{dz}{dt}$ are continuous on $[a,b]$ and not simultaneously 0 on (a,b).
- A curve *C* is **piecewise smooth** if the interval can be partitioned into a finite number of subintervals, on each of which *C* is smooth.

Example 1: Find a piecewise smooth parametrization of the path *C*.

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

$\sin^2 t+\cos^2 t=1$

$\frac{x^2}{16}+\frac{y^2}{9}=1$

$\frac{x^2}{16}=\sin^2 t \qquad \frac{y^2}{9}=\cos^2 t$

$x=4\sin t \qquad y=3\cos t$

$x(t) \qquad y(t)$

on $[0,2\pi)$

$\frac{\partial x}{\partial t}=4\cos t$, $\frac{\partial x}{\partial t}=0$ when $t=\frac{\pi}{2},\frac{3\pi}{2}$

$\frac{\partial y}{\partial t}=-3\sin t$, $\frac{\partial y}{\partial t}=0$ when $t=0,\pi$

So $\frac{\partial x}{\partial t}$ and $\frac{\partial y}{\partial t}$ are continuous on $[0,2\pi)$ and they are not simultaneously zero on $(0,2\pi)$.

So $\vec{r}(t)=4\sin t\,\hat{\imath}+3\cos t\,\hat{\jmath}$, $0\le t\le 2\pi$ is a smooth parametrization of the path C.

DEFINITION OF LINE INTEGRAL

If f is defined in a region containing a smooth curve C of finite length, then the **line integral of f along C** is given by

$$\int_C f(x,y)\,ds = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta s_i \qquad \text{plane}$$

or

$$\int_C f(x,y,z)\,ds = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i \qquad \text{space}$$

provided this limit exists.

*To evaluate a line integral over a plane curve C given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, use the fact that $ds = \|\mathbf{r}'(t)\|\,dt = \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$.

THEOREM: EVALUATION OF A LINE INTEGRAL AS A DEFINITE INTEGRAL

Let f be continuous in a region containing a smooth curve C.

If C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, where $a \le t \le b$, then

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$

If C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where $a \le t \le b$, then

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$$

Note that if $f(x,y,z) = 1$, the line integral gives the arc length of the curve C.

That is, $\int_C 1\,ds = \int_a^b \|r'(t)\|\,dt = \text{length of curve } C$.

Example 2: Evaluate the line integral along the given path.

$$\int_C 8xyz\,ds = \int_0^2 1440t^2 (13)\,dt = \frac{13\cdot \cancel{1440}^{480}}{\cancel{3}} t^3 \Big|_0^2$$

$C:\ \mathbf{r}(t) = 12t\mathbf{i} + 5t\mathbf{j} + 3\mathbf{k}$.

$0 \le t \le 2$

$= 6240(8-0)$

$= \boxed{49920}$

① $f(x,y,z) = 8xyz$

$x(t) = 12t$, $x'(t) = 12$

$y(t) = 5t$, $y'(t) = 5$

$z(t) = 3$, $z'(t) = 0$

② $f(x(t), y(t), z(t))$

$= f(12t, 5t, 3)$

$= 8(12t)(5t)(3)$

$= 1440t^2$

③ $\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$

$= \sqrt{(12)^2 + (5)^2 + 0)^2} = 13$

DEFINITION OF LINE INTEGRAL OF A VECTOR FIELD

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by $\mathbf{r}(t)$, $a \le t \le b$. The <u>**line integral of F on C**</u> is given by

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\,dt$$

Example 3: Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is represented by $\mathbf{r}(t)$

$\mathbf{F}(x,y,z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

$C:\ \mathbf{r}(t) = 2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$0 \le t \le \pi$

①

$x(t) = 2\sin t$, $x'(t) = 2\cos t$

$y(t) = 2\cos t$, $y'(t) = -2\sin t$

$z(t) = \frac{1}{2}t^2$, $z'(t) = t$

② $\vec{r}\,'(t) = 2\cos t\,\hat{i} - 2\sin t\,\hat{j} + t\,\hat{k}$

$\vec{F}(x(t), y(t), z(t))$

$= \vec{F}(2\sin t, 2\cos t, \frac{1}{2}t^2)$

$= (2\sin t)^2\,\hat{i} + (2\cos t)^2\,\hat{j} + (\frac{1}{2}t^2)^2\,\hat{k}$

$= 4\sin^2 t\,\hat{i} + 4\cos^2 t\,\hat{j} + \frac{1}{4}t^4\,\hat{k}$

So $\vec{F}\cdot\vec{r}\,'(t) = (4\sin^2 t)(2\cos t) + (4\cos^2 t)(-2\sin t)$

$+ (\frac{1}{4}t^4)(t)$

$= 8\sin^2 t\cos t - 8\cos^2 t\sin t + \frac{t^5}{4}$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^\pi \left(8\sin^2 t\cos t - 8\cos^2 t\sin t + \frac{t^5}{4}\right) dt$

$= \frac{8}{3}\sin^3 t + \frac{8}{3}\cos^3 t + \frac{t^6}{24}\Big|_0^\pi$

$= \left[\left(0 - \frac{8}{3} + \frac{\pi^6}{24}\right) - \left(0 + \frac{8}{3} + 0\right)\right]$

$= \frac{\pi^6}{24} - \frac{16}{3}$

$= \boxed{\frac{1}{24}(\pi^6 - 128)}$

LINE INTEGRALS IN DIFFERENTIAL FORM

If $\mathbf{F}$ is a vector field of the form $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$, and C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then $\mathbf{F}\cdot d\mathbf{r}$ is often written as $Mdx + Ndy$.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt \\ &= \int_a^b (M\mathbf{i}+N\mathbf{j})\cdot(x'(t)\mathbf{i}+y'(t)\mathbf{j})\,dt \\ &= \int_a^b \left(M\frac{dx}{dt}+N\frac{dy}{dt}\right)dt \\ &= \int_C (Mdx + ndy)\end{aligned}$$

*The parenthesis are often omitted.

Example 4: Evaluate the integral $\int_C (2x-y)\,dx + (x+3y)\,dy$ along the path C

C: arc on $y = x^{3/2}$ from $(0,0)$ to $(4,8)$

Parametrization of C

let $x = t$

$\frac{dx}{dt} = 1 \to dx = dt$

$0 \le t \le 4$

$y = t^{3/2}$

$\frac{dy}{dt} = \frac{3}{2}t^{1/2}$

$dy = \frac{3}{2}t^{1/2}\,dt$

$2x - y = 2t - t^{3/2}$

$x + 3y = t + 3t^{3/2}$

$$\int_0^4 [2t - t^{3/2}]\,dt + (t + 3t^{3/2})\left(\tfrac{3}{2}t^{1/2}\,dt\right)$$

$$= \int_0^4 \left(2t - t^{3/2} + \tfrac{3}{2}t^{3/2} + \tfrac{9}{2}t^2\right)dt$$

$$= \int_0^4 \left(2t + \tfrac{1}{2}t^{3/2} + \tfrac{9}{2}t^2\right)dt$$

$$= \left(t^2 + \tfrac{1}{\cancel{2}}\cdot\tfrac{\cancel{2}}{5}t^{5/2} + \tfrac{3}{2}t^3\right)\Big|_0^4$$

$$= \left(16 + \tfrac{1}{5}(32) + \tfrac{3}{\cancel{2}}(\cancel{64}\,32)\right) - (0)$$

$$= \frac{80 + 32 + 480}{5} = \boxed{\frac{592}{5}}$$