3/9/11

- Warm up
- Lecture 3.6

Friday

Lecture 3.7

Monday

3.8

Friday 3/18

Review

Monday 3/21

Exam 3/ch.13

HW ch.13 is due

↳ 13.2 is extracredit

When you are done with your homework you should be able to…

- π Find and use directional derivatives of a function of two variables
- π Find the gradient of a function of two variables
- π Use the gradient of a function of two variables in applications
- π Find directional derivatives and gradients of functions of three variables

Warm-up: Normalize the following vector (aka find the unit vector):

$$\mathbf{v} = 6\mathbf{i} - \mathbf{j}$$

$\|\vec{v}\| = \sqrt{6^2 + (-1)^2}$

$\|\vec{v}\| = \sqrt{37}$

$\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \boxed{\dfrac{1}{\sqrt{37}}(6\hat{\imath} - \hat{\jmath})}$

Recall that the slope of a surface in the *x*-direction is given by $f_x(x,y)$

And the slope of a surface in the *y*-direction is given by $f_y(x,y)$.

In this section, we will find that these two partial derivatives can be used to find the slope in any direction.

DEFINITION: DIRECTIONAL DERIVATIVE

Let f be a function of two variables x and y and let $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ be a unit vector. Then the **directional derivative of *f* in the direction of u**, denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x,y) = \lim_{t\to 0}\frac{f(x+t\cos\theta, y+t\sin\theta) - f(x,y)}{t}$$

provided the limit exists.

Let $z = f(x,y)$ be a surface and $P(x_0, y_0)$ is a point in the domain of f. The "direction" of the directional derivative is $\vec{u} = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$, θ is the angle made with the positive x-axis.

THEOREM: DIRECTIONAL DERIVATIVE

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ is

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

There are infinitely many directional derivatives to a surface at a given point—one for each direction specified by $\mathbf{u}$.

Example 1: Find the directional derivative of the following functions at the given point and direction.

a. $f(x,y) = x^3 - y^3$, at the point $P(4,3)$, in the direction $\mathbf{v} = \frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})$

$D_{\vec{v}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$

$D_{\vec{v}} f(x,y) = 3x^2\left(\frac{\sqrt{2}}{2}\right) - 3y^2\left(\frac{\sqrt{2}}{2}\right)$

$D_{\vec{v}} f(4,3) = 3(4)^2\left(\frac{\sqrt{2}}{2}\right) - 3(3)^2\left(\frac{\sqrt{2}}{2}\right)$

$= 24\sqrt{2} - \frac{27\sqrt{2}}{2}$

$= \frac{48\sqrt{2} - 27\sqrt{2}}{2} = \boxed{\frac{21\sqrt{2}}{2}}$

$\vec{u} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\vec{v} = \frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j}$

already a unit vector, so

$\cos\theta = \frac{\sqrt{2}}{2}$ and $\sin\theta = \frac{\sqrt{2}}{2}$

b. $f(x,y) = \cos(x+y)$, at the point $P(0,\pi)$, in the direction $Q\left(\frac{\pi}{2},0\right)$

$D_{\vec{u}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$

$D_{\vec{u}} f(x,y) = -\sin(x+y)\left(\frac{1}{\sqrt{5}}\right) - \sin(x+y)\left(-\frac{2}{\sqrt{5}}\right)$

$D_{\vec{u}} f(0,\pi) = \frac{-\sin(0+\pi)}{\sqrt{5}} + \frac{2\sin(0+\pi)}{\sqrt{5}}$

$= \boxed{0}$

$\vec{PQ} = \left\langle \frac{\pi}{2} - 0, 0 - \pi \right\rangle$

$= \left\langle \frac{\pi}{2}, -\pi \right\rangle$

$\|\vec{PQ}\| = \sqrt{\left(\frac{\pi}{2}\right)^2 + (-\pi)^2}$

$= \sqrt{\frac{\pi^2}{4} + \pi^2}$

$= \sqrt{\frac{5\pi^2}{4}}$

$= \frac{\pi\sqrt{5}}{2}$

So $\vec{u} = \frac{\left\langle \frac{\pi}{2}, -\pi \right\rangle}{\frac{\pi\sqrt{5}}{2}}$

$\vec{u} = \frac{\langle 1, -2 \rangle}{\sqrt{5}}$

DEFINITION: GRADIENT OF A FUNCTION OF TWO VARIABLES

Let $z = f(x, y)$, be a function of x and y such that f_x and f_y exist. Then the **gradient of *f***, denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

∇f is read as "del *f*". Another notation for the gradient is $\mathbf{grad}\, f(x, y)$.

Example 2: Find the gradient of $f(x, y) = \ln(x^2 - y)$, at the point $(2,3)$.

$$\nabla f(x,y) = f_x(x,y)\hat{i} + f_y(x,y)\hat{j}$$

$$\nabla f(x,y) = \frac{2x}{x^2-y}\hat{i} + \frac{-1}{x^2-y}\hat{j}$$

$$\nabla f(2,3) = \frac{2(2)}{(2)^2-3}\hat{i} - \frac{1}{(2)^2-3}\hat{j}$$

$$= \boxed{4\hat{i} - \hat{j}}$$

THEOREM: ALTERNATIVE FORM OF THE DIRECTIONAL DERIVATIVE

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Example 3: Use the gradient to find the directional derivative of the function $f(x, y) = \sin 2x \cos y$ at the point $P(0,0)$ in the direction of $Q\left(\frac{\pi}{2}, \pi\right)$.

$$D_{\vec{u}} f(0,0) = \nabla f(0,0) \cdot \vec{u}$$

$$= \langle 2, 0 \rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

$$= (2)\left(\frac{1}{\sqrt{5}}\right) + (0)\left(\frac{2}{\sqrt{5}}\right)$$

$$= \boxed{\frac{2}{\sqrt{5}}}$$

$$\nabla f(x,y) = f_x(x,y)\hat{i} + f_y(x,y)\hat{j}$$

$$\nabla f(x,y) = 2\cos 2x \cos y\,\hat{i} - \sin 2x \sin y\,\hat{j}$$

$$\nabla f(0,0) = 2(1)(1)\hat{i} + 0\hat{j}$$

$$= 2\hat{i}$$

$$\vec{u} = \frac{\left\langle \frac{\pi}{2}, \pi \right\rangle}{\left\| \left\langle \frac{\pi}{2}, \pi \right\rangle \right\|}$$

$$= \frac{\left\langle \frac{\pi}{2}, \pi \right\rangle}{\frac{\pi\sqrt{5}}{2}}$$

$$= \frac{1}{\sqrt{5}}(\hat{i} + 2\hat{j})$$

THEOREM: PROPERTIES OF THE GRADIENT

Let f be differentiable at the point (x, y).

1. If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y) = 0$ for all $\mathbf{u}$
2. The direction of *maximum* increase of f is given by $\nabla f(x, y)$. The maximum value of $D_{\mathbf{u}} f(x, y)$ is $\|\nabla f(x, y)\|$.
3. The direction of *minimum* increase of f is given by $-\nabla f(x, y)$. The minimum value of $D_{\mathbf{u}} f(x, y)$ is $-\|\nabla f(x, y)\|$.

Example 4: The surface of a mountain is modeled by the equation $h(x, y) = 5000 - 0.001x^2 - 0.004y^2$. A mountain climber is at the point $(500, 300, 4390)$. In what direction should the climber move in order to ascend at the greatest rate?

$$\nabla h(x,y) = h_x(x,y)\hat{i} + h_y(x,y)\hat{j}$$

$$\nabla h(x,y) = -.002x\,\hat{i} - .008y\,\hat{j}$$

$$\|\nabla h(x,y)\| = \sqrt{.000004x^2 + .000064y^2}$$

$$\|\nabla h(500,300)\| = 2.6 \rightarrow \text{max value of the gradient}$$

$$\nabla h(500,300) = -.002(500)\hat{i} - .008(300)\hat{j}$$

$$= -\hat{i} - 2.4\hat{j}$$

The climber should move in the direction $-\hat{i} - 2.4\hat{j}$ to ascend at the greatest rate

THEOREM: GRADIENT IS NORMAL TO LEVEL CURVES

If f is differentiable at (x_0, y_0) and $\nabla f(x, y) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).

THEOREM: PROPERTIES OF THE GRADIENT

Let f be a function of x, y, z with continuous first partial derivatives. The **directional derivative of f** in the direction of a unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is given by

$$D_{\mathbf{u}} f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z)$$

The **gradient of f** is defined to be

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

1. $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y, z) = 0$ for all $\mathbf{u}$
3. The direction of *maximum* increase of f is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}} f(x, y, z)$ is $\left\|\nabla f(x, y, z)\right\|$.
4. The direction of *minimum* increase of f is given by $-\nabla f(x, y, z)$. The minimum value of $D_{\mathbf{u}} f(x, y, z)$ is $-\left\|\nabla f(x, y, z)\right\|$.

Example 5: Find the gradient of the function $w = xy^2z^2$ and the maximum value of the directional derivative at the point $(2,1,1)$.

let $w = f(x,y,z)$

$$\nabla f(x,y,z) = f_x(x,y,z)\hat{i} + f_y(x,y,z)\hat{j} + f_z(x,y,z)\hat{k}$$

$$\nabla f(x,y,z) = y^2z^2\hat{i} + 2xyz^2\hat{j} + 2xy^2z\hat{k}$$

$$\nabla f(2,1,1) = (1)^2(1)^2\hat{i} + 2(2)(1)(1)^2\hat{j} + 2(2)(1)^2(1)k$$

$$= \hat{i} + 4\hat{j} + 4\hat{k}$$

max value of the directional derivative at the point $(2,1,1)$ is $\|\nabla f(2,1,1)\| = \sqrt{(1)^2 + (4)^2 + (4)^2} = \boxed{\sqrt{33}}$