3/7/11

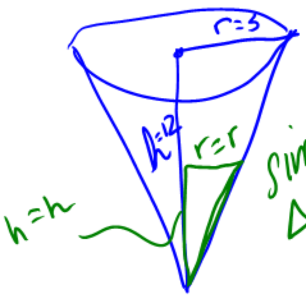
- warmup
- Lecture 13.5

Wednesday	Friday
13.6	13.7

When you are done with your homework you should be able to…

- π Use the chain rules for functions of several variables
- π Find partial derivatives implicitly

Warm-up: A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is eight feet deep.



similar Δ's

$V=\frac{\pi r^2 h}{3} \rightarrow V(h)=\frac{\pi}{3}\left(\frac{5}{12}h\right)^2\cdot h$

$\frac{5}{12}=\frac{r}{h}$

$r=\frac{5}{12}h$

$\frac{d}{dt}\left(V(h)\right)=\frac{d}{dt}\left[\frac{\pi}{3}\cdot\frac{25}{144}h^3\right]$

$\frac{dV}{dt}=\frac{\pi}{\cancel{3}}\cdot\frac{25}{144}\cancel{3}h^2\frac{dh}{dt}$

$\frac{10\,ft^3}{min}=\frac{\pi\cdot 25}{144}(8\,ft)^2\left(\frac{dh}{dt}\right)$

$\frac{\cancel{10}\,ft^{\cancel{3}}\cdot \cancel{144}}{\cancel{25}\cdot\cancel{64}\,\cancel{ft^2}\cdot\pi\,min}=\frac{dh}{dt}$

$\frac{9}{10\pi}\,ft/min=\frac{dh}{dt}$

$\frac{dh}{dt}\approx 0.2865\,ft/min$

When full, $r=\frac{10}{2}=5$ and $h=12$

$\frac{dV}{dt}=10$

want $\frac{dh}{dt}$ when $h=8$

The rate of change of the depth of the water when the water is eight feet deep is approx. 0.2865 ft/min

THEOREM: CHAIN RULE: ONE INDEPENDENT VARIABLE

Let $w=f(x,y)$, where f is a differentiable function x and y. If $x=g(t)$ and $y=h(t)$, where g and h are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}$$

This can be extended to any number of variables. If $w=f(x_1,x_2,\ldots,x_n)$, you would have

$$\frac{dw}{dt}=\frac{\partial w}{\partial x_1}\frac{dx_1}{dt}+\frac{\partial w}{\partial x_2}\frac{dx_2}{dt}+\cdots+\frac{\partial w}{\partial x_n}\frac{dx_n}{dt}$$

Example 1: Find $\frac{dw}{dt}$ (a) using the appropriate chain rule and (b) by converting *w* to a function of *t* before differentiating.

a. $w=\cos(x-y),\ \ x=t^2,\ \ y=1$

converting to function of t first

$\frac{d}{dt}w = \frac{d}{dt}\cos(t^2-1)$

$\frac{dw}{dt} = -\sin(t^2-1)\cdot 2t$

$\boxed{\frac{dw}{dt} = -2t\sin(t^2-1)}$

using appropriate chain rule

$w=f(x,y)$, $x=t^2$, $y=1$

$\frac{dx}{dt}=2t$, $\frac{dy}{dt}=0$

$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$

$\frac{dw}{dt} = -\sin(x-y)\cdot 2t + [\sin(x-y)\cdot(-1)]\cdot 0$

$\frac{dw}{dt} = -2t\sin(x-y) + 0$

$\frac{dw}{dt} = -2t\sin(x-y) \rightarrow \boxed{\frac{dw}{dt} = -2t\sin(t^2-1)}$

b. $w=xyz,\ \ x=t^2,\ \ y=2t,\ \ z=e^{-t}$

old way

$w=(t^2)(2t)(e^{-t})$

$\frac{d}{dt}w = \frac{d}{dt}(2t^3e^{-t})$

$\frac{dw}{dt} = 6t^2e^{-t} - 2t^3e^{-t}$

$\boxed{\frac{dw}{dt} = 2t^2e^{-t}(3-t)}$

New way

$x=t^2$, $y=2t$, $z=e^{-t}$

$\frac{dx}{dt}=2t$, $\frac{dy}{dt}=2$, $\frac{dz}{dt}=-e^{-t}$

$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$

$\frac{dw}{dt} = yz(2t) + xz(2) + xy(-e^{-t})$

$\frac{dw}{dt} = 2te^{-t}(2t) + t^2e^{-t}(2) + t^2\cdot 2t(-e^{-t})$

$\frac{dw}{dt} = 2t^2e^{-t}(2+1-t)$

$\boxed{\frac{dw}{dt} = 2t^2e^{-t}(3-t)}$

THEOREM: CHAIN RULE: ~~ONE~~ TWO INDEPENDENT VARIABLES

Let $w=f(x,y)$, where f is a differentiable function x and y. If $x=g(s,t)$ and $y=h(s,t)$, such that the first partials $\partial x/\partial s$, $\partial x/\partial t$, $\partial y/\partial s$, and $\partial y/\partial t$ all exist, then $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ exist and are given by

$$\frac{dw}{ds}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

This can be extended to any number of variables. If w is a differentiable function of the n variables where each $x_1,x_2,\ldots,x_n$ is a differentiable function of the m variables $t_1,t_2,\ldots,t_m$, then for $w=f(x_1,x_2,\ldots,x_n)$, you would have

$$\frac{\partial w}{\partial t_1}=\frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1}+\frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_1}+\cdots+\frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_1}$$

$$\frac{\partial w}{\partial t_2}=\frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_2}+\frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_2}+\cdots+\frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_2}$$

$$\vdots$$

$$\frac{\partial w}{\partial t_m}=\frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_m}+\frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_m}+\cdots+\frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m}$$

Example 2: Find $\partial w/\partial s$ and $\partial w/\partial t$ using the appropriate chain rule, and evaluate each partial derivative at the given values of s and t.

Function	Point
$w=y^3-3x^2y$	$s=0,\quad t=1$
$x=e^s,\quad y=e^t$	

$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial w}{\partial s}=-6xy(e^s)+(3y^2-3x^2)(0)$$

$$\frac{\partial w}{\partial s}=-6e^s\cdot e^t\cdot e^s$$

$$\frac{\partial w}{\partial s}=-6e^{2s+t}\Big|_{(0,1)}=-6e^{0+1}=\boxed{-6e}$$

$$\frac{\partial w}{\partial t}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$\frac{\partial w}{\partial t}=(-6xy)(0)+(3y^2-3x^2)\cdot e^t$$

$$\frac{\partial w}{\partial t}=3(e^{2t}-e^{2s})e^t\Big|_{(0,1)}=3(e^2-1)e$$

$$=\boxed{3e(e^2-1)}$$

THEOREM: CHAIN RULE: IMPLICIT DIFFERENTIATION

If the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \quad F_y(x, y) \neq 0.$$

If the equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{dz}{dx} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \text{ and } \frac{dz}{dy} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \quad F_z(x, y, z) \neq 0.$$

Example 3: Differentiate implicitly to find $\frac{dy}{dx}$.

$$\cos x + \tan xy + 5 = 0$$

$F(x,y) = 0$ so $F(x,y) = \cos x + \tan xy + 5$

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}$$

$$\frac{dy}{dx} = -\frac{-\sin x + y\sec^2(xy)}{x\sec^2(xy)}$$

$$\boxed{\frac{dy}{dx} = \frac{\sin x - y\sec^2 xy}{x\sec^2 xy}}$$

Example 4: Differentiate implicitly to find the first partial derivatives of z.

$$x\ln y+y^2z+z^2=8$$

$$x\ln y+y^2z+z^2-8=0$$

$$F(x,y,z)=x\ln y+y^2z+z^2-8$$

$$\frac{\partial z}{\partial x}=-\frac{F_x(x,y,z)}{F_z(x,y,z)}$$

$$\boxed{\frac{\partial z}{\partial x}=-\frac{\ln y}{y^2+2z}}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y(x,y,z)}{F_z(x,y,z)}$$

$$\frac{\partial z}{\partial y}=-\frac{\frac{x}{y}+2yz\cdot\frac{y}{y}}{y^2+2z}$$

$$\boxed{\frac{\partial z}{\partial y}=-\frac{x+2y^2z}{y(y^2+2z)}}$$

Example 5: The radius of a right circular cone is increasing at a rate of 6 inches per minute, and the height is decreasing at a rate of 4 inches per minute. What are the rates of change of the volume and surface area when the radius is 12 inches and the height is 36 inches?

We know: $\frac{dr}{dt}=6\text{in}/\text{min}$

$\frac{dh}{dt}=-4\text{in}/\text{min}$

$$V=\frac{\pi r^2h}{3}$$

$$\frac{dV}{dt}=\frac{\pi}{3}\left[\frac{\partial V}{\partial r}\cdot\frac{dr}{dt}+\frac{\partial V}{\partial h}\cdot\frac{dh}{dt}\right]$$

$$\frac{dV}{dt}=\frac{\pi}{3}\left[2rh\frac{dr}{dt}+r^2\cdot\frac{dh}{dt}\right]$$

$$\frac{dV}{dt}=\frac{\pi}{3}\left[2(12)(36)(6)+(12)^2\cdot(-4)\right]$$

$$\frac{dV}{dt}=\frac{1536\pi\text{ in}^3}{\text{min}}$$

$$S=\pi r\sqrt{r^2+h^2}+\pi r^2$$

$$\frac{dS}{dt}=\pi\left[\frac{\partial S}{\partial r}\frac{dr}{dt}+\frac{\partial S}{\partial h}\cdot\frac{dh}{dt}\right]$$

$$\frac{dS}{dt}=\pi\left[\left(\sqrt{r^2+h^2}+r\cdot\frac{\cancel{2}r}{\cancel{2}\sqrt{r^2+h^2}}+2r\right)\cdot\frac{dr}{dt}+r\cdot\frac{\cancel{2}h}{\cancel{2}\sqrt{r^2+h^2}}\cdot\frac{dh}{dt}\right]$$

$$\frac{dS}{dt}=\pi\left[\left(\sqrt{12^2+36^2}+\frac{12\cdot12}{\sqrt{12^2+36^2}}\right)(6)+\frac{12\cdot36}{\sqrt{12^2+36^2}}(-4)\right]$$

$$\boxed{\frac{dS}{dt}\approx656.6\text{ in}^2/\text{min}}$$