3/16/11

- warm up
- lecture 13.9

Friday

Review

Monday

Exam 3/Ch. 13

Ch13 HW due

(13.2 extra credit)

3/23 and 3/25 there is no class. You will be given an out-of-class assignment on Monday

When you are done with your homework you should be able to…

- π Solve optimization problems involving functions of several variables
- π Use the method of least squares

Warm-up: Examine the function $g(x,y) = 120x + 120y - xy - x^2 - y^2$ for relative extrema and saddle points.

$g_x(x,y) = 120 - y - 2x$

$0 = 120 - y - 2x$

$2x + y = 120$

$g_y(x,y) = 120 - x - 2y$

$0 = 120 - x - 2y$

$x + 2y = 120$

$2x + y = 120$

$x + 2y = 120$

$2x + y = x + 2y$

$x = y$

$2x + x = 120$

$3x = 120$

$x = 40$

$y = 40$

critical point is $(40,40)$

$g_{xx}(x,y) = -2 = g_{xx}(40,40)$

$g_{yy}(x,y) = -2 = g_{yy}(40,40)$

$g_{xy}(x,y) = -1 = g_{xy}(40,40)$

$d = (-2)(-2) - (-1)^2$

$d = 3 > 0$ and

$g_{xx}(40,40) < 0$

So there's a rel max at $(40, 40, 4800)$

Example 1: Find the minimum distance from the point $(1,2,3)$ to the plane $2x + 3y + z = 12$. (HINT: To simplify the computations, minimize the square of the distance).

Step1: Analysis

Let S be the square of the distance from $(1,2,3)$ to a point on the plane (x,y,z)

$S = (x-1)^2 + (y-2)^2 + (z-3)^2$

Step2: Primary equation

$S = (x-1)^2 + (y-2)^2 + (z-3)^2$

Step3: Reduce to a $f(x,y)$

$2x + 3y + z = 12 \rightarrow z = 12 - 2x - 3y$

$S(x,y) = (x-1)^2 + (y-2)^2 + [(12-2x-3y) - 3]^2$

$S(x,y) = (x-1)^2 + (y-2)^2 + (9-2x-3y)^2$

Step4: Optimize

$S_x(x,y) = 2(x-1) - 4(9-2x-3y)$

$0 = (x-1) - 2(9-2x-3y) = 5x + 6y - 19$

$S_y(x,y) = 2(y-2) - 6(9-2x-3y)$

$0 = (y-2) - 3(9-2x-3y) = 6x + 10y - 29$

$\begin{cases} 5x + 6y = 19 \\ 6x + 10y = 29 \end{cases}$

$-30x - 36y = -114$

$30x + 50y = 145$

$14y = 31$

$y = \frac{31}{14}$

$$5x + 6y = 19$$

$$5x + \cancel{6}^{3}\left(\frac{31}{\cancel{14}_{7}}\right) = 19$$

$$5x + \frac{93}{7} = 19$$

$$5x = \frac{133}{7} - \frac{93}{7}$$

$$5x = \frac{40}{7}$$

$$x = \frac{8}{7}$$

$$S\left(8/7, 31/14\right) = \left(8/7 - 1\right)^2 + \left(31/14 - 2\right)^2 + \left(9 - 2\left(8/7\right) - 3\left(31/14\right)\right)^2$$

$$S\left(8/7, 31/14\right) = \left(\frac{1}{7}\right)^2 + \left(\frac{3}{14}\right)^2 + \left(\frac{\overbrace{126 - 32 - 93}^{1}}{14}\right)^2$$

$$S\left(8/7, 31/14\right) = \frac{1}{49} + \frac{9}{196} + \frac{1}{196}$$

$$S\left(8/7, 31/14\right) = \frac{14}{196} \Rightarrow \sqrt{S\left(8/7, 31/14\right)} = \sqrt{\frac{14}{196}} = \frac{\sqrt{14}}{14}$$

Step 5: Conclusion

$\boxed{\text{The minimum distance from the point } (1,2,3) \text{ to the plane } 2x + 3y + z = 12 \text{ is } \frac{\sqrt{14}}{14} \text{ units.}}$

Example 2: Find three positive numbers *x, y,* and *z* which have a sum of 1 and the sum of the squares is a minimum.

Step 1: Analysis

Let x be the 1st positive #
Let y be the 2nd " #
Let z be the 3rd " #

$$x+y+z=1$$
$$z=1-x-y$$

Step 2: Primary Equation

$$S(x,y,z)=x^2+y^2+z^2$$

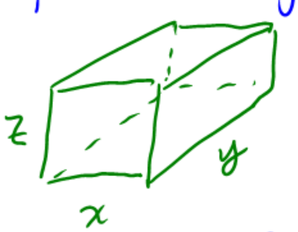
Step 3: Reduce primary

$$S(x,y)=x^2+y^2+(1-x-y)^2$$

Example 3: The material for constructing the base of an open box costs 1.5 times as much per unit area as the material for constructing the sides. For a fixed amount of money *C*, find the dimensions of the box of largest volume that can be made.

Step 1: Analysis

Let x, y, and z be the length, width & height



$$C_0=1.5(xy)+2(yz)+2(xz)$$
$$C_0=1.5xy+z(2x+2y)$$
$$\frac{C_0-1.5xy}{2(x+y)}=z$$

Step 2 and 3: Primary Equation & reduce primary

$$V(x,y,z)=xyz$$
$$V(x,y)=xy\frac{(C_0-1.5xy)}{2(x+y)}$$
$$V(x,y)=\frac{C_0xy-1.5x^2y^2}{2(x+y)}$$

Step 4: Optimize

$$V_x(x,y)=\frac{1}{2}\left[\frac{(C_0y-3xy^2)(x+y)-(C_0xy-1.5x^2y^2)(1)}{(x+y)^2}\right]$$

$$V_x(x,y)=\frac{1}{2}\left(\frac{\cancel{C_0xy}-3x^2y^2+C_0y^2-3xy^3-\cancel{C_0xy}+1.5x^2y^2}{(x+y)^2}\right)$$

$$V_x(x,y) = \frac{y^2(C_0 - 1.5x^2 - 3xy)}{2(x+y)^2}$$

$$V_y(x,y) = \frac{1}{2}\left[\frac{(C_0x - 3x^2y)(x+y) - (C_0xy - 1.5x^2y^2)(1)}{(x+y)^2}\right]$$

$$V_y(x,y) = \frac{1}{2}\left(\frac{C_0x^2 - 3x^3y + \cancel{C_0xy} - 3x^2y^2 - \cancel{C_0xy} + 1.5x^2y^2}{(x+y)^2}\right)$$

$$V_y(x,y) = \frac{x^2(C_0 - 3xy - 1.5y^2)}{2(x+y)^2}$$

$$x^2(C_0 - 3xy - 1.5y^2) = y^2(C_0 - 3xy - 1.5x^2)$$

* only way to be equivalent is if $x = y$

$$0 = x^2(C_0 - 3x(x) - 1.5(x)^2)$$

$$x^2 = 0 \quad \text{or} \quad C_0 - \frac{9}{2}x^2 = 0$$

$$\cancel{x=0} \quad \text{or} \quad C_0 = \frac{9}{2}x^2$$

$$\sqrt{\frac{2C_0}{9}} = \sqrt{x^2}$$

$$x = \frac{\sqrt{2C_0}}{3} = y$$

$$z = \frac{C_0 - 1.5\left(\frac{\sqrt{2C_0}}{3}\right)\left(\frac{\sqrt{2C_0}}{3}\right)}{2\left(\frac{\sqrt{2C_0}}{3} + \frac{\sqrt{2C_0}}{3}\right)} = \frac{\frac{3C_0}{\cancel{3}} - \frac{\overset{1}{\cancel{3}}C_0}{9\cancel{3}}}{2 \cdot \frac{2\sqrt{2C_0}}{\cancel{3}}} = \frac{2C_0}{4\sqrt{2C_0}} = \boxed{\frac{C_0}{2\sqrt{2C_0}}}$$

Example 4: A retail outlet sells two types of riding lawn mowers, the prices of which are p_1 and p_2. Find p_1 and p_2, so as to maximize total revenue, where $R = 515p_1 + 805p_2 + 1.5p_1p_2 - 1.5p_1^2 - p_2^2$.

THEOREM: LEAST SQUARES REGRESSION LINE

The **least squares regression line** for $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_n, y_n)\}$ is given by $f(x) = ax + b$, where

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \quad \text{and} \quad b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i\right)$$

Example 5: Find the least squares regression line for the points $(1,0)$, $(3,3)$, $(5,6)$.