3/11/11

- warm up
- Lecture 13.7

Monday	Next Friday
13.8	Review
Wednesday 13.9 *Review 3.7	Monday, 3/21 Exam 3/Ch. 13

Find the angle of inclination θ of the tangent plane to the surface at the given point

(48) $2xy - z^3 = 0, \quad (2,2,2)$

$F(x,y,z) = 2xy - z^3$

$\nabla F(x,y,z) = 2y\,\hat{i} + 2x\,\hat{j} - 3z^2\,\hat{k}$

$\nabla F(2,2,2) = 4\,\hat{i} + 4\,\hat{j} - 12\,\hat{k}$

THE ANGLE INCLINATION OF A PLANE

$$\cos\theta = \frac{|\mathbf{n}\cdot\mathbf{k}|}{\|\mathbf{n}\|} \qquad 0 \le \theta \le \frac{\pi}{2}$$

$$\cos\theta = \frac{|\nabla F(2,2,2)\cdot\hat{k}|}{\|\nabla F(2,2,2)\|}$$

$$\cos\theta = \frac{|\langle 4,4,-12\rangle \cdot \langle 0,0,1\rangle|}{|4|\sqrt{1^2+1^2+(-3)^2}}$$

$$\cos\theta = \frac{|-12|}{4\sqrt{11}}$$

$$\cos\theta = \frac{3}{\sqrt{11}}$$

$$\theta = \arccos\frac{3}{\sqrt{11}}$$

$\theta \approx 0.4405$

or $\boxed{\theta \approx 25.24^\circ}$

When you are done with your homework you should be able to…

- π Find equations of tangent planes and normal lines to surfaces
- π Find the angle of inclination of a plane in space
- π Compare the gradients $\nabla f(x,y)$ and $\nabla F(x,y)$

Warm-up: Find the general equation of the plane containing the points (2, 1, 1), (0, 4, 1), and (-2, 1, 4).

P, Q, R

$\vec{u} = \overrightarrow{QP} = \langle 2, -3, 0 \rangle$

$\vec{v} = \overrightarrow{QR} = \langle -2, -3, 3 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 0 \\ -2 & -3 & 3 \end{vmatrix}$$

$= (-9-0)\hat{i} - (6-0)\hat{j} + (-6-6)\hat{k}$

$= -9\hat{i} - 6\hat{j} - 12\hat{k}$

$= -3\langle 3, 2, 4 \rangle$

So $a=3, b=2, c=4$

$3(x-2)+2(y-1)+4(z-1)=0$

General form:

$3x-6+2y-2+4z-4=0$

$\boxed{3x+2y+4z=12}$

DEFINITION OF TANGENT PLANE AND NORMAL LINE

Let F be differentiable at the point $P(x_0, y_0, z_0)$ on the surface given by $F(x,y,z)=0$ such that $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$.

1. The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the **tangent plane to *S* at *P***.
2. The line through P having the direction of $\nabla F(x_0, y_0, z_0)$ is called the **normal line to *S* at *P***.

We've been using $z=f(x,y)$ for a surface S.

Rewrite as $F(x,y,z)=f(x,y)-z$, S is the level surface of F given by $F(x,y,z)=0$

Example 1: Find a unit normal vector to the surface at the given point. (*HINT:* normalize the gradient vector $\nabla F(x,y,z)$).

$$x^2+y^2+z^2=11, \text{ at the point } P(3,1,1)$$

$F(x,y,z) = x^2+y^2+z^2-11$

$\nabla F(x,y,z) = 2x\hat{i}+2y\hat{j}+2z\hat{k}$

$\|\nabla F(x,y,z)\| = \sqrt{(2)^2x^2+2^2y^2+2^2z^2}$

$= \sqrt{3(2)^2}$

$= 2\sqrt{x^2+y^2+z^2}$

$\dfrac{\cancel{2}(x\hat{i}+y\hat{j}+z\hat{k})}{\cancel{2}\sqrt{x^2+y^2+z^2}}$

$= \dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$

at $(3,1,1)$:

$= \boxed{\dfrac{3\hat{i}+\hat{j}+\hat{k}}{\sqrt{11}}}$

THEOREM: EQUATION OF TANGENT PLANE

If F is differentiable at (x_0,y_0,z_0), then an equation of the tangent plane to the surface is given by $F(x,y,z)=0$ at (x_0,y_0,z_0) is

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

Example 2: Find an equation of the tangent plane to the surface at the given point.

$$h(x, y) = \ln\sqrt{x^2 + y^2}, \text{ at the point } P(3, 4, \ln 5)$$

$$z = \frac{1}{2}\ln(x^2+y^2)$$

$$H(x,y,z) = \frac{1}{2}\ln(x^2+y^2) - z$$

$$\nabla H(x,y,z) = \frac{1}{2}\cdot\frac{2x}{x^2+y^2}\hat{i} + \frac{1}{2}\cdot\frac{2y}{x^2+y^2}\hat{j} - \hat{k}$$

$$\nabla H(3,4,\ln 5) = \frac{3}{(3)^2+(4)^2}\hat{i} + \frac{4}{(3)^2+(4)^2}\hat{j} - \hat{k} = \frac{3}{25}\hat{i} + \frac{4}{25}\hat{j} - \hat{k}$$

$$\frac{3}{25}(x-3) + \frac{4}{25}(y-4) - (z - \ln 5) = 0$$

$$\boxed{3(x-3) + 4(y-4) - 25(z-\ln 5) = 0}$$

Example 3: Find an equation of the tangent plane and find symmetric equations of the normal line to the surface at the given point.

$u = \frac{y}{x} = yx^{-1}$

$$z = \arctan\frac{y}{x}, \text{ at the point } \left(1, 1, \frac{\pi}{4}\right)$$

$$F(x,y,z) = \arctan\left(\frac{y}{x}\right) - z$$

$$\nabla F(x,y,z) = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}\hat{i} + \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\hat{j} - \hat{k}$$

$$\nabla F(1,1,\pi/4) = \frac{\frac{-1}{1^2}}{1+\left(\frac{1}{1}\right)^2}\hat{i} + \frac{\frac{1}{1}}{1+\left(\frac{1}{1}\right)^2}\hat{j} - \hat{k} = \left\langle -\frac{1}{2}, \frac{1}{2}, -1 \right\rangle$$

Equation of tangent plane to the surface:

$$2\cdot\left[-\frac{1}{2}(x-1) + \frac{1}{2}(y-1) - (z - \pi/4)\right] = [0]\cdot 2$$

$$\boxed{-(x-1) + (y-1) - 2(z - \pi/4) = 0}$$

Symmetric equations of the normal line to the surface:

$$\boxed{\frac{x-1}{-1} = \frac{y-1}{1} = \frac{z - \pi/4}{-2}}$$

Example 4: Find the path of a heat-seeking particle placed at the point in space $(2,2,5)$ with a temperature field $T(x,y,z)=100-3x-y-z^2$.

THE ANGLE INCLINATION OF A PLANE

$$\cos\theta = \frac{|\mathbf{n}\cdot\mathbf{k}|}{\|\mathbf{n}\|}$$

THEOREM: GRADIENT IS NORMAL TO LEVEL SURFACES

If F is differentiable at (x_0,y_0,z_0) and $\nabla F(x_0,y_0,z_0)\neq\mathbf{0}$, then $\nabla F(x_0,y_0,z_0)$ is normal to the level surface through (x_0,y_0,z_0).