2/9/11

- warm up using 12.3 worksheet
- Lecture 12.3

Friday

12.4

Monday

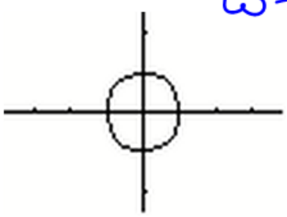
12.5

When you are done with your homework you should be able to…

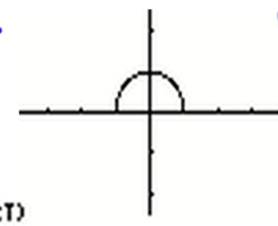
- π Describe the velocity and acceleration associated with a vector-valued function
- π Use a vector-valued function to analyze projectile motion

Warm-up: Consider the circle given by $\mathbf{r}(t) = (\cos \omega t)\mathbf{i} + (\sin \omega t)\mathbf{j}$. Use a graphing calculator in parametric mode to graph this circle for several values of ω.

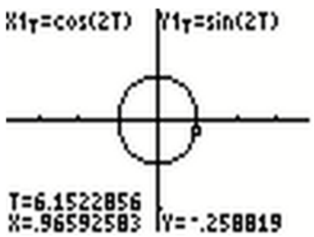
How does ω affect the velocity of the terminal point as it traces out the curve?



$\omega = 1$, traced once $[0, 2\pi)$



made it halfway $[0, 2\pi)$



made it around twice $[0, 2\pi)$

As ω increases, the velocity of the terminal point increases

For a given value of ω, does the speed appear constant?

yes

Does the acceleration appear constant?

NO → since there's in direction, there must be a change in acceleration

DEFINITIONS OF VELOCITY AND ACCELERATION

If x and y are twice differentiable functions of t, and $\mathbf{r}$ is a vector-valued function given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then the velocity vector, acceleration vector, and speed at time t are as follows:

Velocity $= \mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$

Acceleration $= \mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j}$

Speed $= \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$

For motion along a space curve, the definitions are as follows:

Velocity $= \mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$

Acceleration $= \mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$

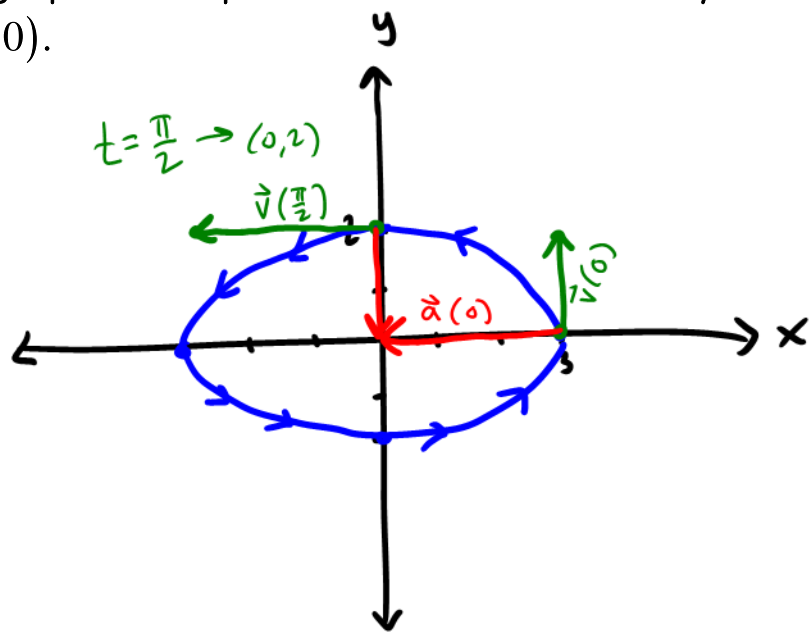
Speed $= \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$

Example 1: The position vector $\mathbf{r}(t) = 3\cos t\mathbf{i} + 2\sin t\mathbf{j}$ describes the path of an object moving in the xy-plane. Sketch a graph of the path and sketch the velocity and acceleration vectors at the point $(3,0)$.

$x = 3\cos t$ $y = 2\sin t$

$\cos^2 t + \sin^2 t = 1$

$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$



$t = ?$ when $x = 3$ and $y = 0$

$t = 0$

$x(t) = 3\cos t$ $y(t) = 2\sin t$

$x'(t) = -3\sin t$ $y'(t) = 2\cos t$

$\vec{v}(t) = -3\sin t\,\hat{\imath} + 2\cos t\,\hat{\jmath}$

$\vec{v}(0) = 2\hat{\jmath}$

$\vec{a}(t) = -3\cos t\,\hat{\imath} - 2\sin t\,\hat{\jmath}$

$\vec{a}(0) = -3\hat{\imath}$

Example 2: The position vector $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + 2t^{3/2}\mathbf{k}$ describes the path of an object moving in space. Find the velocity, speed and acceleration of the object.

$$\vec{v}(t) = \boxed{2t\hat{i} + \hat{j} + 3t^{1/2}\hat{k}}$$

$$\vec{a}(t) = \boxed{2\hat{i} + \tfrac{3}{2}t^{-1/2}\hat{k}}$$

$$\text{speed} = \|\vec{v}(t)\|$$
$$= \sqrt{(2t)^2 + (1)^2 + (3t^{1/2})^2}$$
$$= \sqrt{4t^2 + 1 + 9t}$$
$$= \boxed{\sqrt{4t^2 + 9t + 1}}$$

THEOREM: POSITION FUNCTION FOR A PROJECTILE

Neglecting air resistance, the path of a projectile launched from an initial height h with initial speed v_0 and angle of elevation θ is described by the vector function

$$\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$$

where g is the gravitational constant.

$g = 32 \text{ ft/sec}^2$ OR
$g = 9.8 \text{ m/sec}^2$

Example 3: Determine the maximum height and range of a projectile fired at a height 3 feet above the ground with an initial velocity of 900 feet per second and at an angle of 45° above the horizontal.

$h = 3$
$v_0 = 900$
$\theta = 45°$
$g = 32$

$$\vec{r}(t) = (900\cos 45°)t\hat{i} + \left[3 + (900\sin 45°)t - \tfrac{1}{2}(32)t^2\right]\hat{j}$$
$$\vec{r}(t) = 450\sqrt{2}\,t\hat{i} + \left[3 + 450\sqrt{2}\,t - 16t^2\right]\hat{j}$$

So $x(t) = 450\sqrt{2}\,t$ and $y(t) = 3 + 450\sqrt{2}\,t - 16t^2$

Max height:

$$y'(t) = 450\sqrt{2} - 32t$$

$$0 = 450\sqrt{2} - 32t$$

$$32t = 450\sqrt{2}$$

$$t = \frac{225\sqrt{2}}{16}$$

$$y\left(\frac{225\sqrt{2}}{16}\right) = 3 + 450\sqrt{2}\left(\frac{225\sqrt{2}}{16}\right) - 16\left(\frac{225\sqrt{2}}{16}\right)^2$$

$$\approx \boxed{6331.125\text{ ft}}$$

Set $y(t)=0$ to find max range

$$0 = 3 + 450\sqrt{2}\,t - 16t^2$$

$$x(t) = 450\sqrt{2}\,t$$

$$x(39.779) \approx 450\sqrt{2}\,(39.779)$$

$$= \boxed{25{,}315.500\text{ ft}}$$

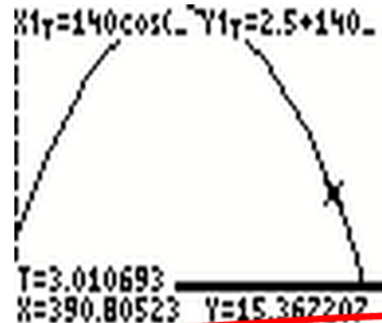
Example 4: A baseball is hit from a height of 2.5 feet above the ground with an initial velocity of 140 feet per second and at an angle of 22° above the horizontal. Use a graphing utility to graph the path of the ball and determine whether it will clear a ten foot high fence located 375 feet from home plate.

$h=2.5,\ g=32,\ \theta=22^\circ,\ v_0=140$

$x(t)=140\cos 22^\circ t$

$y(t)=2.5+140\sin 22^\circ t-\frac{1}{2}\cdot 32t^2$

$y(t)=2.5+140\sin 22^\circ t-16t^2$



yes → $x=390.8,\ y=15.4$ and the graph is ↓.

Example 5: Find the maximum speed of a point on the circumference of an automobile tire of radius one foot when the automobile is traveling at 55 mph. Compare this speed with the speed of the automobile. Use the following formula for the cycloid:

$\mathbf{r}(t)=b(\omega t-\sin\omega t)\mathbf{i}+b(1-\cos\omega t)\mathbf{j}$

ω is the constant angular velocity of the circle and

b is the radius of the circle.

$\vec{v}(t)=b(\omega-\omega\cos\omega t)\hat{i}+b(\omega\sin\omega t)\hat{j}$

$\vec{v}(t)=b\omega\left[(1-\cos\omega t)\hat{i}+\sin\omega t\,\hat{j}\right]$

Speed $=\|\vec{v}(t)\|=|b\omega|\sqrt{(1-\cos\omega t)^2+\sin^2\omega t}$

$=b\omega\sqrt{1-2\cos\omega t+\underbrace{\cos^2\omega t+\sin^2\omega t}_{1}}$

$=b\omega\sqrt{2}\sqrt{1-\cos\omega t}$

max value occurs when $\omega t=\pi, 3\pi, 5\pi, \ldots$

$\sqrt{2}\,b\omega\cdot\sqrt{2}=2b\omega$

$\frac{55\ \cancel{\text{miles}}}{\cancel{\text{hr}}}\times\frac{5280\text{ft}}{1\ \cancel{\text{mile}}}\times\frac{1\ \cancel{\text{hr}}}{3600\text{sec}}$

$\approx 80.67\ \text{ft/sec}$

$=80.67\ \text{rad/sec}$

recall $v=r\omega$ and $r=1$

So max speed of a point on the tire is $2\cdot b\cdot\omega=2\cdot 1\cdot\omega$

→ $=2\omega=2(80.67)\ \text{ft/sec}$

→ $\boxed{110\text{ mph}}$