2/7/11

- Warmup using 12.2 WS
- Lecture 12.2

Wednesday
12.3

Friday
12.4

Exam 2 is 2/23/11

power reducing formulas aka double ∡ identity for cosine function

$$\cos 2A = 1 - 2\sin^2 A$$

$$\sin^2 A = \frac{1-\cos 2A}{2}$$

OR

$$\cos 2A = 2\cos^2 A - 1$$

$$\cos^2 A = \frac{1+\cos 2A}{2}$$

When you are done with your homework you should be able to…

π Differentiate a vector-valued function

π Integrate a vector-valued function

Warm-up 1: Evaluate the following derivatives with respect to x.

1. $\frac{\partial}{\partial x} y = \frac{\partial}{\partial x} \frac{\sin^2 3x}{\sqrt{x}}$

$$\frac{dy}{dx} = \frac{\left[2(\sin 3x)^1(\cos 3x)(3)\right](x^{1/2}) - \sin^2 3x\left(\frac{1}{2}x^{-1/2}\right)}{(\sqrt{x})^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}x^{-1/2}\left(12x^1 \sin 3x \cos 3x - \sin^2 3x\right)}{x}$$

$$\boxed{\frac{dy}{dx} = \frac{12x\sin 3x\cos 3x - \sin^2 3x}{2x^{3/2}}}$$

2. $\frac{\partial}{\partial x} f(x) = \frac{\partial}{\partial x} xe^{-2x}$

$$f'(x) = 1e^{-2x} + x\left(-2e^{-2x}\right)$$

$$\boxed{f'(x) = \frac{1-2x}{e^{2x}}}$$

3. $y = \ln\left(\frac{5x}{e^{x^2}}\right)^{2/3} - \frac{6}{x} - \arctan 3x^3$

$$y = \frac{2}{3}\ln 5x - \frac{2}{3}\ln e^{x^2} - 6x^{-1} - \arctan(3x^3)$$

$$\frac{\partial}{\partial x} y = \frac{\partial}{\partial x}\frac{2}{3}\ln 5x - \frac{\partial}{\partial x}\frac{2}{3}x^2 - \frac{\partial}{\partial x}6x^{-1} - \frac{\partial}{\partial x}\arctan(3x^3)$$

$u = 3x^3$
$u' = 9x^2$

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{\cancel{5}}{\cancel{5}x}\right) - \frac{2}{3}(2x) - 6\left(-x^{-2}\right) - \frac{9x^2}{1+9x^6}$$

$$\boxed{\frac{dy}{dx} = \frac{2}{3x} - \frac{4x}{3} + \frac{6}{x^2} - \frac{9x^2}{1+9x^6}}$$

Warm-up 2: Integrate.

1. $\int\left(6x^2 - \sin^2 3x\right)dx$

$$= \frac{6x^3}{3} - \frac{1}{2}\int(1-\cos 6x)\,dx$$

$$= 2x^3 - \frac{1}{2}\left(x - \frac{\sin 6x}{6}\right) + C$$

$$= \boxed{2x^3 - \frac{x}{2} + \frac{1}{12}\sin 6x + C}$$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du$

2. $\int \frac{\sqrt{\ln x}}{x} dx = \int \frac{u^{1/2}}{\cancel{x}}(\cancel{x}du)$

$= \int u^{1/2} du$

$= \frac{u^{3/2}}{\frac{3}{2}} + C$

$= \boxed{\frac{2}{3}(\ln x)^{3/2} + C}$

3. $\int \frac{4}{\sqrt{1-x^2}} dx = \boxed{4\arcsin x + C}$

DEFINITION OF THE DERIVATIVE OF A VECTOR-VALUED FUNCTION

The **derivative of a vector-valued function $\mathbf{r}$** is defined by

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

for all t for which the limit exists. If $\mathbf{r}'(c)$ exists, then $\mathbf{r}$ is **differentiable at c**. If $\mathbf{r}'(c)$ exists for all c in an open interval I then $\mathbf{r}$ is **differentiable on the open interval I**. Differentiability of vector-valued functions can be extended to closed intervals by considering one-sided limits.

Other notation: $\mathbf{r}'(t)$, $\frac{d}{dt}[\mathbf{r}(t)]$, $D_t[\mathbf{r}(t)]$, $\frac{d\mathbf{r}}{dt}$

THEOREM: DIFFERENTIATION OF VECTOR-VALUED FUNCTIONS

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are differentiable functions of *t*, then $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$. provided *f* and *g* have limits as $t \to a$.

2. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions of *t*, then $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$.

Higher-order derivatives of vector-valued functions are obtained by successive differentiation of each component function.

The **parametrization of the curve** represented by the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is **smooth on an open interval *I*** if f', g', and h' are continuous on *I* and $\mathbf{r}'(t) \neq \mathbf{0}$ for any value of *t* on the open interval *I*.

THEOREM: PROPERTIES OF THE DERIVATIVE

Let $\mathbf{r}$ and $\mathbf{u}$ be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

1. $D_t\left[c\mathbf{r}(t)\right] = c\mathbf{r}'(t)$
2. $D_t\left[\mathbf{r}(t) \pm \mathbf{u}(t)\right] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
3. $D_t\left[f(t)\mathbf{u}(t)\right] = f(t)\mathbf{u}'(t) + f'(t)\mathbf{u}(t)$
4. $D_t\left[\mathbf{r}(t) \cdot \mathbf{u}(t)\right] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$
5. $D_t\left[\mathbf{r}(t) \times \mathbf{u}(t)\right] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$
6. $D_t\left[\mathbf{r}(f(t))\right] = \mathbf{r}'(f(t)) \cdot f'(t)$
7. If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$

Example 1: Find $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$.

$\frac{\partial}{\partial t}\mathbf{r}(t) = \frac{\partial}{\partial t}(t^2 + t)\mathbf{i} + \frac{\partial}{\partial t}(t^2 - t)\mathbf{j}$

$\vec{r}\,'(t) = (2t+1)\hat{\imath} + (2t-1)\hat{\jmath} \rightarrow \langle 2t+1, 2t-1 \rangle$

$\vec{r}\,''(t) = 2\hat{\imath} + 2\hat{\jmath} \rightarrow \langle 2, 2 \rangle$

$$\begin{aligned}
\vec{r}\,'(t) \cdot \vec{r}\,''(t) &= (2t+1)(2) + (2t-1)(2) \\
&= 2(2t+1+2t-1) \\
&= 2(4t) \\
&= \boxed{8t}
\end{aligned}$$

Example 2: Find $D_t\left[\mathbf{r}(t)\times\mathbf{u}(t)\right]$

$\mathbf{r}(t)=t\mathbf{i}+2\sin t\mathbf{j}+2\cos t\mathbf{k},$

$\mathbf{u}(t)=\frac{1}{t}\mathbf{i}+2\sin t\mathbf{j}+2\cos t\mathbf{k},$

$$\vec{r}(t)\times\vec{u}(t)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ t & 2\sin t & 2\cos t\\ t^{-1} & 2\sin t & 2\cos t\end{vmatrix}$$

$$=(4\sin t\cos t-4\cos t\sin t)\hat{i}-(2t\cos t-2t^{-1}\cos t)\hat{j}+(2t\sin t-2t^{-1}\sin t)\hat{k}$$

$$=2\left[(t^{-1}\cos t-t\cos t)\hat{j}+(t\sin t-t^{-1}\sin t)\hat{k}\right]$$

$$=2\left[\cos t(t^{-1}-t)\hat{j}+\sin t(t-t^{-1})\hat{k}\right]$$

$$\frac{d}{dt}\left[\vec{r}(t)\times\vec{u}(t)\right]=2\left\{\left[-\sin t(t^{-1}-t)+\cos t(-t^{-2}-1)\right]\hat{j}+\left[\cos t(t-t^{-1})+\sin t\cdot(1+t^{-2})\right]\hat{k}\right\}$$

DEFINITION OF INTEGRATION OF VECTOR-VALUED FUNCTIONS

1. If $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}$, where f and g are continuous on $[a,b]$ then the <u>**indefinite integral (antiderivative)**</u> of $\mathbf{r}$ is

$$\int\mathbf{r}(t)\,dt=\left[\int f(t)\,dt\right]\mathbf{i}+\left[\int g(t)\,dt\right]\mathbf{j}+\vec{C}$$

and its <u>**definite integral**</u> over the interval $a\le t\le b$ is

$$\int_a^b\mathbf{r}(t)\,dt=\left[\int_a^b f(t)\,dt\right]\mathbf{i}+\left[\int_a^b g(t)\,dt\right]\mathbf{j}$$

2. If $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$, where f, g, and h are continuous on $[a,b]$ then the <u>**indefinite integral (antiderivative)**</u> of $\mathbf{r}$ is

$$\int\mathbf{r}(t)\,dt=\left[\int f(t)\,dt\right]\mathbf{i}+\left[\int g(t)\,dt\right]\mathbf{j}+\left[\int h(t)\,dt\right]\mathbf{k}+\vec{C}$$

and its <u>**definite integral**</u> over the interval $a\le t\le b$ is

$$\int_a^b\mathbf{r}(t)\,dt=\left[\int_a^b f(t)\,dt\right]\mathbf{i}+\left[\int_a^b g(t)\,dt\right]\mathbf{j}+\left[\int_a^b g(t)\,dt\right]\mathbf{k}$$

Example 3: Evaluate the indefinite integral

$$\int \left(4t^3\mathbf{i} + 6t\mathbf{j} - 4\sqrt{t}\mathbf{k}\right)dt = \left(\frac{\cancel{4}t^4}{\cancel{4}} + C_1\right)\hat{i} + \left(\frac{\cancel{6}t^2}{\cancel{2}} + C_2\right)\hat{j} - \left(\frac{4t^{3/2}}{3/2} + C_3\right)\hat{k}$$

$$= \boxed{t^4\hat{i} + 3t^2\hat{j} - \frac{8}{3}t^{3/2}\hat{k} + \vec{C}}$$

Example 4: Evaluate the definite integral

$$\int_0^{\pi/4} \left[\sec t \tan t\mathbf{i} + \tan t\mathbf{j} + 2\sin t\cos t\mathbf{k}\right]dt = \left(\sec t\Big|_0^{\pi/4}\hat{i} - \left(\ln|\cos t|\Big|_0^{\pi/4}\hat{j}\right.\right.$$

$$+ \int 2u\cancel{\cos t}\left(\frac{du}{\cancel{\cos t}}\right)$$

$$= \sec t\Big|_0^{\pi/4}\hat{i} - \ln|\cos t|\Big|_0^{\pi/4}\hat{j} + \frac{\cancel{2}\sin^2 t}{\cancel{2}}\Big|_0^{\pi/4}\hat{k}$$

$$= \left(\sqrt{2} - 1\right)\hat{i} - \left(\ln\left|\frac{\sqrt{2}}{2}\right| - \ln|1|\right)\hat{j} + \left[\left(\frac{1}{\sqrt{2}}\right)^2 - 0\right]\hat{k}$$

$$= \boxed{\left(\sqrt{2} - 1\right)\hat{i} - \ln\left(\frac{\sqrt{2}}{2}\right)\hat{j} + \frac{1}{2}\hat{k}}$$

$u = \sin t$

$\frac{du}{dt} = \cos t$

$dt = \frac{du}{\cos t}$

or

$2\sin t\cos t$

$= \sin 2t$