

2/16/11

Lecture 12.5

wednesday, 2/23

Have 12.1-12.5
homework done

so you know
what questions
to ask for the
review

Friday to next Monday

Holiday → Do NOT KILL ANY
BRAIN CELLS!

Friday, 2/25/11 is exam 2

See 2/9/11 notes for the last
example on 12.3 worksheet

When you are done with your homework you should be able to...

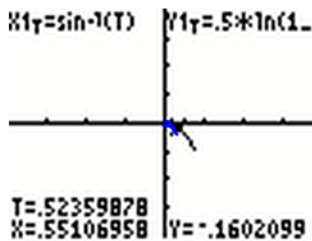
- π Find the arc length of a space curve
- π Use the arc length parameter to describe a plane curve or space curve
- π Find the curvature of a curve at a point on the curve
- π Use a vector-valued function to find frictional force

Warm-up: Find the arc length of the curve

$x = \arcsin t$ and $y = \ln \sqrt{1-t^2}$ on the interval $\left[0, \frac{1}{2}\right]$.
 $y = \frac{1}{2} \ln(1-t^2)$

$$\left(\frac{dx}{dt}\right)^2 = \left(\frac{1}{\sqrt{1-t^2}}\right)^2 \quad \left(\frac{dy}{dt}\right)^2 = \left[\frac{1}{2} \left(\frac{-2t}{1-t^2}\right)\right]^2$$

$$= \frac{1}{1-t^2} \quad = \frac{t^2}{(1-t^2)^2}$$



p. 396 in 9th ed Larson

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left| \frac{a+u}{a-u} \right| + C$$

$a=1$
 $u=t$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$s = \int_0^{1/2} \sqrt{\frac{1}{1-t^2} + \frac{t^2}{(1-t^2)^2}} dt$$

$$s = \int_0^{1/2} \frac{\sqrt{1-t^2+t^2}}{(1-t^2)^2} dt$$

$$s = \int_0^{1/2} \frac{1}{1-t^2} dt$$

$$s = \left. \left(\frac{1}{2(1)}\right) \ln \left| \frac{1+t}{1-t} \right| \right|_0^{1/2}$$

$$s = \frac{1}{2} \left[\ln \left| \frac{1+\frac{1}{2}}{1-\frac{1}{2}} \right| - \ln \left| \frac{1+0}{1-0} \right| \right]$$

$$s = \frac{1}{2} (\ln 3 - 0)$$

$$s = \ln \sqrt{3} \text{ units}$$

THEOREM: ARC LENGTH OF A SPACE CURVE

If C is a smooth curve given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ on an interval $[a, b]$, then the arc length of C is

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt = \int_a^b \|\mathbf{r}'(t)\| dt$$

Example 1: Find the arc length of the curve given by $\mathbf{r}(t) = 2\sin t\mathbf{i} + 5t\mathbf{j} + 2\cos t\mathbf{k}$ over $[0, \pi]$.

$$x(t) = 2\sin t \quad y(t) = 5t \quad z(t) = 2\cos t$$

$$x'(t) = 2\cos t \quad y'(t) = 5 \quad z'(t) = -2\sin t$$

$$s = \int_0^\pi \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

$$s = \int_0^\pi \sqrt{4\cos^2 t + 25 + 4\sin^2 t} dt$$

$$s = \int_0^\pi \sqrt{4(\cos^2 t + \sin^2 t) + 25} dt$$

$$s = \int_0^\pi \sqrt{29} dt$$

$$s = \sqrt{29}t \Big|_0^\pi$$

$s = \sqrt{29} \pi$ units

* For motion along a curve the convenient parameter is time t . For studying geometric properties of a curve,

DEFINITION: ARC LENGTH FUNCTION

Let C be a smooth curve given by $\mathbf{r}(t)$ defined on the closed interval $[a, b]$, then the arc length of C is

$$s(t) = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2} du = \int_a^t \|\mathbf{r}'(u)\| du$$

The arc length s is called the arc length parameter. The arc length function is nonnegative as it measures the distance along C from the initial point. Using the definition of the arc length function and the second fundamental theorem of

calculus, you can conclude $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$.

↳ the convenient parameter is the arc length, s .

Example 2: Find the arc length function for the line segment given by $\mathbf{r}(t) = (3-3t)\mathbf{i} + 4t\mathbf{j}$, $0 \leq t \leq 1$, and write \mathbf{r} as a function of the parameter s .

$$\vec{r}'(t) = -3\hat{i} + 4\hat{j}$$

$$\|\vec{r}'(t)\| = \sqrt{9+16} = 5$$

$$s(t) = \int_0^t 5 du$$

$$s(t) = 5u \Big|_0^t$$

$$s(t) = 5t$$

$$s = 5t$$

$$t = \frac{s}{5}$$

$$\vec{r}(s) = (3 - 3(\frac{s}{5}))\hat{i} + 4(\frac{s}{5})\hat{j}$$

$$\vec{r}(s) = (3 - \frac{3s}{5})\hat{i} + \frac{4s}{5}\hat{j}$$

$$0 \leq s \leq 5$$

$$0 \leq t \leq 1$$

$$5 \cdot 0 \leq \frac{5 \cdot s}{5} \leq 5 \cdot 1$$

THEOREM: ARC LENGTH PARAMETER

If C is a smooth curve given by

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} \text{ or } \mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k}$$

where s is the arc length parameter, then

$$\|\mathbf{r}'(s)\| = 1.$$

Moreover, if t is any parameter for the vector-valued function \mathbf{r} such that

$$\|\mathbf{r}'(s)\| = 1, \text{ then } t \text{ must be the arc length parameter.}$$

DEFINITION OF CURVATURE

Let C be a smooth curve (in the plane or in space) given by $\mathbf{r}(s)$, where s is the arc length parameter. The curvature K at s is given by

$$K = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{T}'(s)\|$$

Example 3: Find the curvature K of the curve, where s is the arc length parameter.

$$\mathbf{r}(s) = (3+s)\mathbf{i} + \mathbf{j}$$

$$\hat{\mathbf{T}}(s) = \frac{\hat{\mathbf{r}}'(s)}{\|\hat{\mathbf{r}}'(s)\|} = \frac{\langle 1, 0 \rangle}{\sqrt{1^2 + 0^2}} = \hat{\mathbf{i}}$$

$$K = \|\hat{\mathbf{T}}'(s)\|$$

$$K = \|\vec{0}\|$$

$$K = 0$$

$$\hat{\mathbf{T}}'(s) = \vec{0}$$

THEOREM: FORMULAS FOR CURVATURE

If C is a smooth curve given by $\mathbf{r}(t)$, then the curvature K of C at t is given by

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

easier for this example

Example 4: Find the curvature K of the curve $\mathbf{r}(t) = 2t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$.

$$\vec{r}'(t) = 4t\hat{i} + \hat{j} + t\hat{k} \rightarrow \langle 4t, 1, t \rangle$$

$$\vec{r}''(t) = 4\hat{i} + \hat{k} \rightarrow \langle 4, 0, 1 \rangle$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4t & 1 & t \\ 4 & 0 & 1 \end{vmatrix} = 4(t\hat{j} - \hat{k}) + 1(\hat{i} - 4t\hat{j})$$

$$= 4t\hat{j} - 4\hat{k} + \hat{i} - 4t\hat{j}$$

$$= \hat{i} - 4\hat{k} = \vec{r}'(t) \times \vec{r}''(t)$$

$$\|\hat{i} - 4\hat{k}\| = \sqrt{17}$$

$$\|\vec{r}'(t)\| = \sqrt{17t^2 + 1}$$

$$K = \frac{\sqrt{17}}{(17t^2 + 1)^{3/2}}$$

THEOREM: CURVATURE IN RECTANGULAR COORDINATES

If C is the graph of a twice differentiable function given by $y = f(x)$, then the curvature K at the point (x, y) is given by

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}}$$

Related Stuff: Let C be a curve with curvature K at point P . The circle passing through point P with radius $r = \frac{1}{K}$ is called the **circle of curvature** if the circle lies on the concave side of the curve and shares a common tangent line with the curve at point P . The radius is called the **radius of curvature** at P and the center of the circle is called the **center of curvature**.

Example 4: Find the **curvature** and **radius of curvature** of the plane curve

$y = 2x + \frac{4}{x}$ at $x = 1$.

$$y' = 2 - \frac{4}{x^2}$$

$$y'' = \frac{8}{x^3}$$

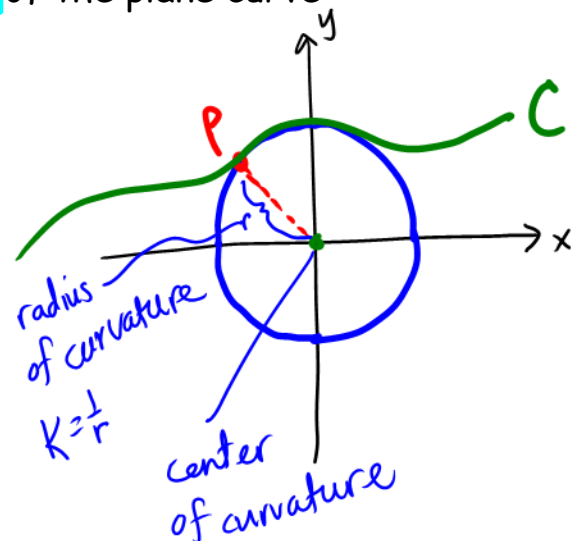
at $x = 1, y' = -2, y'' = 8$

$$K = \frac{|y''|}{(1 + (y')^2)^{3/2}}$$

$$K = \frac{8}{[1 + 4]^{3/2}}$$

$$K = \frac{8}{5^{3/2}}$$

$$r = \frac{1}{K} = \frac{5^{3/2}}{8}$$



THEOREM: ACCELERATION, SPEED, AND CURVATURE

If $\mathbf{r}(t)$ is the position vector for a smooth curve \mathcal{C} then the acceleration vector is given by

$$\mathbf{a}(t) = \frac{d^2s}{dt^2} \mathbf{T} + K \left(\frac{ds}{dt} \right)^2 \mathbf{N}$$

where K is the curvature of \mathcal{C} and $\frac{ds}{dt}$ is the speed.

Frictional Force

A moving object with mass m is in contact with a stationary object. The total force required to produce an acceleration \mathbf{a} along a given path is

$$\begin{aligned} \mathbf{F} &= m\mathbf{a} \\ &= m \left(\frac{d^2s}{dt^2} \right) \mathbf{T} + mK \left(\frac{ds}{dt} \right)^2 \mathbf{N} \\ &= ma_T \mathbf{T} + ma_N \mathbf{N} \end{aligned}$$

Example 5: A 6400-pound vehicle is driven at a speed of 35 mph on a circular interchange of radius 250 feet. To keep the vehicle from skidding off course, what frictional force must the road surface exert on the tires?

$$m = 6400 \text{ lbs}, \quad \frac{\partial s}{\partial t} = 35 \text{ mph}, \quad K = \frac{1}{r} = \frac{1}{250 \text{ ft}}$$

$$\text{frictional force: } mK \left(\frac{\partial s}{\partial t} \right)^2 = (6400 \text{ lbs}) \left(\frac{1}{250 \text{ ft}} \right) \left(\frac{154 \text{ ft}}{3 \text{ sec}} \right)^2$$

$$\frac{35 \text{ miles}}{\text{hr}} \times \frac{5280 \text{ ft}}{1 \text{ mile}} \times \frac{1 \text{ hr}}{3600 \text{ sec}} = \frac{154}{3} \text{ ft/sec}$$

$$\approx 67458.8 \text{ ft-pounds per sec}^2$$