1/21/11

- Warm up using 11.4 worksheet
- Lecture 11.4

Monday

11.5

When you are done with your homework you should be able to…

π Find the cross product of two vectors in space

π Use the triple scalar product of three vectors in space

Warm-up: Find the direction cosines of $\mathbf{u} = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and demonstrate that the sum of the squares of the direction cosines is 1.

$\|\vec{u}\| = \sqrt{25+9+1}$

$\|u\| = \sqrt{35}$

$\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{u} = \langle 5, 3, -1 \rangle$

$$\boxed{\begin{aligned} \cos\alpha &= \frac{u_1}{\|\vec{u}\|} = \frac{5}{\sqrt{35}} \\ \cos\beta &= \frac{u_2}{\|\vec{u}\|} = \frac{3}{\sqrt{35}} \\ \cos\gamma &= \frac{u_3}{\|\vec{u}\|} = -\frac{1}{\sqrt{35}} \end{aligned}}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma \stackrel{?}{=} 1$$

$$\left(\frac{5}{\sqrt{35}}\right)^2 + \left(\frac{3}{\sqrt{35}}\right)^2 + \left(\frac{-1}{\sqrt{35}}\right)^2 \stackrel{?}{=} 1$$

$$\frac{25+9+1}{35} \stackrel{?}{=} 1$$

$$1 = 1 \checkmark$$

DEFINITION OF CROSS PRODUCT OF TWO VECTORS IN SPACE

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in space.

The cross product of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} \cancel{+} (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}.$$

$$\vec{u} \times \vec{v} = \begin{array}{ccc} + & - & + \end{array}$$
$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k}$$

$$= (u_2 v_3 - u_3 v_2)\hat{i} - (u_1 v_3 - u_3 v_1)\hat{j} + (u_1 v_2 - u_2 v_1)\hat{k}$$

THEOREM: ALGEBRAIC PROPERTIES OF THE CROSS PRODUCT

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in space and let c be a scalar.

1. $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$
2. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = (\mathbf{u}\times\mathbf{v})+(\mathbf{u}\times\mathbf{w})$
3. $c(\mathbf{u}\times\mathbf{v}) = (c\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(c\mathbf{v})$
4. $\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \mathbf{0}$
5. $\mathbf{u}\times\mathbf{u} = \mathbf{0}$
6. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$

THEOREM: GEOMETRIC PROPERTIES OF THE CROSS PRODUCT

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in space, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

1. $\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
2. $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$.
3. $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.
4. $\|\mathbf{u}\times\mathbf{v}\|$ = the area of parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

Example 1: Find $\mathbf{u}\times\mathbf{v}$ and show that it is orthogonal to both $\mathbf{u} = \langle -1,1,2\rangle$ and $\mathbf{v} = \langle 0,1,0\rangle$.

$\langle u_1, u_2, u_3\rangle$ $\langle v_1, v_2, v_3\rangle$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 2 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 0 - \begin{vmatrix} \hat{i} & \hat{k} \\ -1 & 2 \end{vmatrix} + 0$$

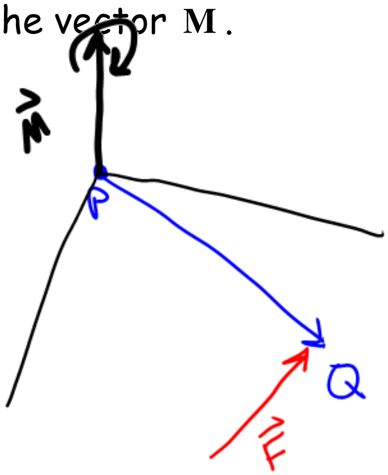
$$= \cancel{-(2\hat{i} - (-\hat{k}))}$$

$$= \boxed{-2\hat{i} - \hat{k}}$$

$$\vec{u}\cdot(\vec{u}\times\vec{v}) = \langle -1,1,2\rangle\cdot\langle -2,0,-1\rangle$$
$$= (-1)(-2)+(1)(0)+(2)(-1)$$
$$= 0 \checkmark$$

$$\vec{v}\cdot(\vec{u}\times\vec{v}) = \langle 0,1,0\rangle\cdot\langle -2,0,-1\rangle$$
$$= 0$$
$$= 0 \checkmark$$

In physics, the cross product can be used to measure **torque**, which is the moment $\mathbf{M}$ of a force $\mathbf{F}$ about a point P. If the point of application of the force is Q, the moment of $\mathbf{F}$ about P is given by $\mathbf{M} = \overrightarrow{PQ} \times \mathbf{F}$. The magnitude of the moment $\mathbf{M}$ measures the tendency of the vector $\overrightarrow{PQ}$ to rotate counterclockwise about an axis directed along the vector $\mathbf{M}$.



Example 2: Both the magnitude and direction of the force on a crankshaft change as the crankshaft rotates. Find the torque on the crankshaft using the position and data shown in the figure.

$\overrightarrow{PQ} = .16\hat{k}$

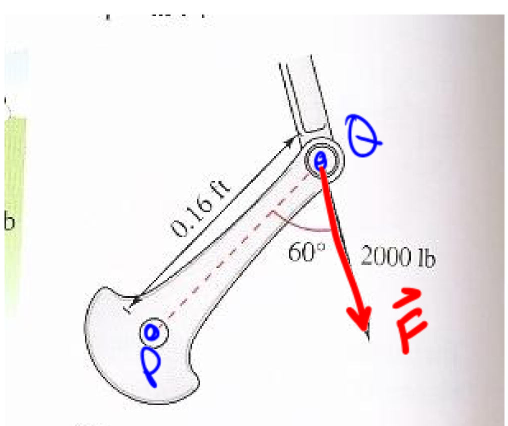
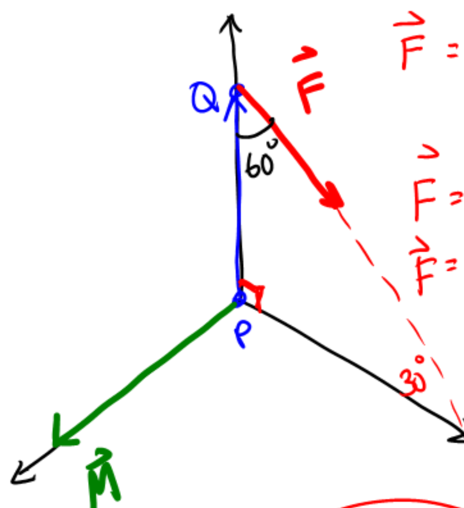


Figure for 38



$\vec{F} = -2000(\cos 30^\circ \hat{j} + \sin 30^\circ \hat{k})$

$\vec{F} = -2000\left(\frac{\sqrt{3}}{2}\hat{j} + \frac{1}{2}\hat{k}\right)$

$\vec{F} = -1000(\sqrt{3}\hat{j} + \hat{k})$

$\vec{F} = -1000$

$$\vec{M} = \overrightarrow{PQ} \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & .16 \\ 0 & -1000\sqrt{3} & -1000 \end{vmatrix} = (0 + 160\sqrt{3})\hat{i}$$

$$= \boxed{160\sqrt{3} \text{ ft-lbs}}$$

THEOREM: THE TRIPLE SCALAR PRODUCT

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$,
The triple scalar product is given by

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

* volume of a parallelepiped with vectors $\vec{u}, \vec{v}$, and $\vec{w}$ as adjacent edges is given by

$$V = |\vec{u}\cdot(\vec{v}\times\vec{w})|$$

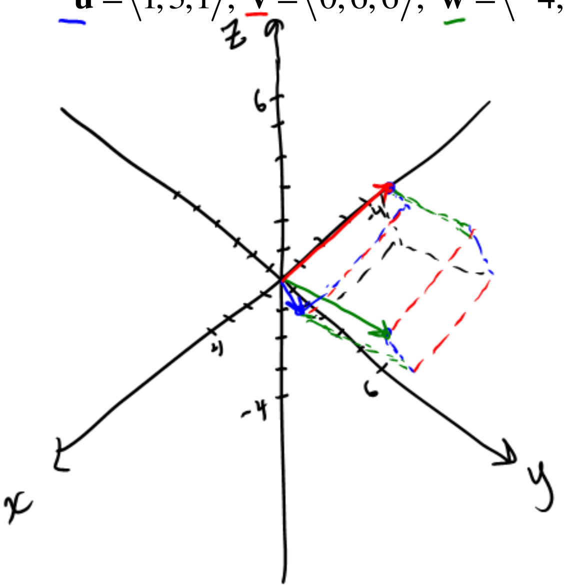
Example 3: Find $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$. $\mathbf{u} = \langle 1,1,1\rangle$, $\mathbf{v} = \langle 2,1,0\rangle$, $\mathbf{w} = \langle 0,0,1\rangle$.

$$u\cdot(v\times w) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 0 - 0 + \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}\cdot 1$$

$$= 1 - 2 = \boxed{-1}$$

Example 4: Find the volume of the parallelepiped having adjacent edges
$\mathbf{u} = \langle 1,3,1\rangle$, $\mathbf{v} = \langle 0,6,6\rangle$, $\mathbf{w} = \langle -4,0,-4\rangle$.



$$V = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 6 & 6 \\ -4 & 0 & -4 \end{vmatrix}$$

$$V = \left| \begin{vmatrix} 6 & 6 \\ 0 & -4 \end{vmatrix} 1 - 0 + \begin{vmatrix} 3 & 1 \\ 6 & 6 \end{vmatrix}(-4) \right|$$

$$V = |(-24-0)(1) + (18-6)(-4)| \quad (12)(-4)$$

$$V = |-72| \rightarrow \boxed{V = 72 \text{ cubic units}}$$