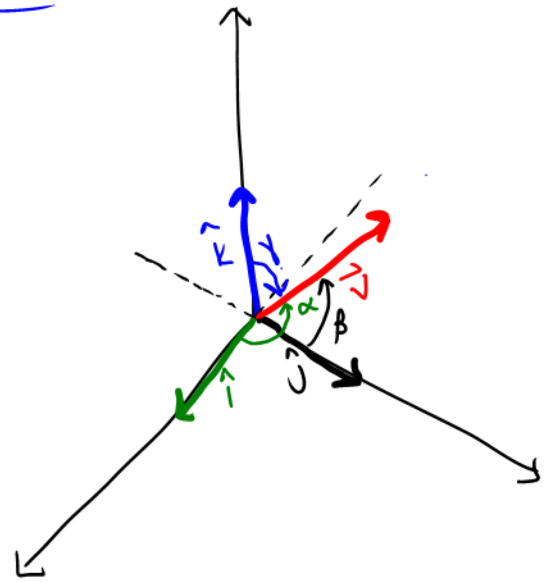
1/19/11

- Warm up
- Lecture 11.3

Friday

- Lecture 11.4



When you are done with your homework you should be able to…

- π Use properties of the dot product of two vectors
- π Find the angle between two vectors using the dot product
- π Find the direction cosines of a vector in space
- π Find the projection of a vector onto another vector
- π Use vectors to find the work done by a constant force

Warm-up: Write the equation of the sphere in standard form. Find the center and the radius

$$9x^2+9y^2+9z^2-6x+18y+1=0$$

$$9x^2-6x \quad +9y^2+18y \quad +9z^2 = -1$$

$$9\left(x^2-\frac{2}{3}x+\left(-\frac{1}{3}\right)^2\right)+9\left(y^2+2y+(1)^2\right)+9z^2=-1+1+9$$

$$\frac{9}{9}\left(x-\frac{1}{3}\right)^2+\frac{9}{9}(y+1)^2+\frac{9}{9}z^2=\frac{9}{9}$$

$$\boxed{\left(x-\frac{1}{3}\right)^2+(y+1)^2+z^2=1}$$

center: $\left(\frac{1}{3},-1,0\right)$

radius: 1

DEFINITION OF DOT PRODUCT (aka inner product aka scalar product)

The dot product of $\mathbf{u}=\langle u_1,u_2\rangle$ and $\mathbf{v}=\langle v_1,v_2\rangle$ is

$$\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2.$$

The dot product of $\mathbf{u}=\langle u_1,u_2,u_3\rangle$ and $\mathbf{v}=\langle v_1,v_2,v_3\rangle$ is

$$\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2+u_3v_3.$$

THEOREM: PROPERTIES OF THE DOT PRODUCT

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in the plane or in space and let c be a scalar.

1. *Commutative Property.* $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$
2. *Distributive Property.* $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$
3. $c(\mathbf{u}\cdot\mathbf{v})=c\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot c\mathbf{v}$
4. $\mathbf{0}\cdot\mathbf{v}=0$
5. $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2$

Example 1: Given $\mathbf{u}=\langle -4,6\rangle$, $\mathbf{v}=\langle 3,7\rangle$ and $\mathbf{w}=\langle 9,-5\rangle$, find each of the following:

a) $\mathbf{u}\cdot\mathbf{w}$ $=(-4)(9)+(6)(-5)$

$=\boxed{-66}$

b) $5\mathbf{u}\cdot\mathbf{v}$

$5\vec{u}=5\langle -4,6\rangle$

$=\langle -20,30\rangle$

$\therefore\ 5\vec{u}\cdot\vec{v}=(-20\cdot 3)+(30\cdot 7)$

$=-60+210$

$=\boxed{150}$

c) $\mathbf{u}\cdot\mathbf{u}$

$\langle -4,6\rangle\cdot\langle -4,6\rangle = 16+36=\boxed{52}$

or $\left(\sqrt{(-4)^2+(6)^2}\right)^2=(\sqrt{52})^2=\boxed{52}$

d) $(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w}$

$\langle -4,6\rangle\cdot\langle 3,7\rangle$

$(-4)3+(6)7=30$

$30\cdot\vec{w}=\boxed{30\langle 9,-5\rangle=\langle 270,-150\rangle}$

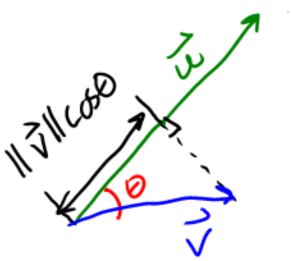
THEOREM: ANGLE BETWEEN TWO VECTORS

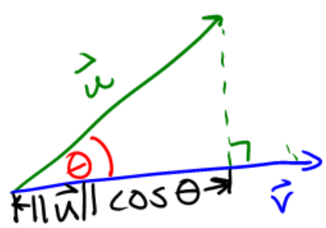
If $\theta,\ 0\le\theta\le\pi,$ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ then

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}.$$

$\vec{u}\cdot\vec{v}=\|\vec{u}\|\left[\|v\|\cos\theta\right]$ is the scalar component of vector $\vec{v}$ along the direction of vector $\vec{u}$ and

$\vec{u}\cdot\vec{v}=\|\vec{v}\|\left[\|\vec{u}\|\cos\theta\right]$ is the scalar component of vector $\vec{u}$ along the direction of vector $\vec{v}$.





Example 2: Find the angle θ between the vectors $\mathbf{u}=3\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and $\mathbf{v}=2\mathbf{i}-3\mathbf{j}$..

$$\cos\theta=\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

$$\cos\theta=\frac{(3)(2)+(2)(-3)+(1)(0)}{\sqrt{(3)^2+(2)^2+(1)^2}\sqrt{(2)^2+(-3)^2}}$$

$$\cos\theta=0$$

$$\boxed{\theta=\frac{\pi}{2}}$$

DEFINITION: ORTHOGONAL VECTORS

The vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if

$$\mathbf{u}\cdot\mathbf{v}=0.$$

Example 3: Determine whether vectors $\mathbf{u}=-2\mathbf{i}+3\mathbf{j}-\mathbf{k}$ and $\mathbf{v}=2\mathbf{i}+\mathbf{j}-\mathbf{k}$ are orthogonal, parallel or neither.

$\vec{u}\cdot\vec{v}=(-2)(2)+(3)(1)+(-1)(-1)$

$\vec{u}\cdot\vec{v}=0$

$\boxed{\vec{u} \text{ and } \vec{v} \text{ are orthogonal.}}$

DIRECTION COSINES

For a vector in the *plane*, we often measure its direction in terms of the angle measured counterclockwise from the positive x-axis to the vector.

In *space*, it is more convenient to measure direction in terms of the angles between the nonzero vector $\mathbf{v}$ and the three unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. The angles α, β and γ are the **direction angles** of $\mathbf{v}$ and $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the **direction cosines** of $\mathbf{v}$.

Activity:

1. Use the theorem for the angle between two vectors to find an alternate form of the dot product. Substitute the unit vector $\mathbf{i}$ for vector $\mathbf{u}$.

$$\cos\alpha=\frac{\hat{i}\cdot\vec{v}}{\|\hat{i}\|\|\vec{v}\|}\Rightarrow\cos\alpha=\frac{\hat{i}\cdot\vec{v}}{1\cdot\|\vec{v}\|}\Rightarrow\hat{i}\cdot\vec{v}=\|\vec{v}\|\cos\alpha$$

2. Now find $\mathbf{v}\cdot\mathbf{i}$ using the component form of each vector.

$$\vec{v}\cdot\hat{\imath} = \langle v_1, v_2, v_3\rangle\cdot\langle 1,0,0\rangle$$
$$\vec{v}\cdot\hat{\imath} = v_1$$

3. Equate your results from parts 1 and 2 and then isolate $\cos\alpha$.

$$v_1 = \|\vec{v}\|\cos\alpha \Rightarrow \cos\alpha = \frac{v_1}{\|\vec{v}\|}$$

4. Repeat this exercise to find $\cos\beta$ and $\cos\gamma$.

$$\cos\beta = \frac{\hat{\jmath}\cdot\vec{v}}{\|\hat{\jmath}\|\|\vec{v}\|}$$

$$\cos\beta = \frac{\langle 0,1,0\rangle\cdot\langle v_1, v_2, v_3\rangle}{1\cdot\|\vec{v}\|}$$

$\Rightarrow \cos\beta = \dfrac{v_2}{\|v\|}$

Similarly, $\cos\gamma = \dfrac{v_3}{\|v\|}$

5. Find the normalized form of any nonzero vector $\mathbf{v}$, that is, find two expressions for $\frac{\mathbf{v}}{\|\mathbf{v}\|}$, using your previous results.

$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{v_1}{\|\vec{v}\|}\hat{\imath} + \frac{v_2}{\|\vec{v}\|}\hat{\jmath} + \frac{v_3}{\|\vec{v}\|}\hat{k}$$

or $\dfrac{\vec{v}}{\|\vec{v}\|} = \cos\alpha\,\hat{\imath} + \cos\beta\,\hat{\jmath} + \cos\gamma\,\hat{k}$

6. Find $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$. Hint: $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector.

$\boxed{1}$

Example 4: Find the direction angles of the vector $\mathbf{u} = -4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$.

$\|\vec{u}\| = \sqrt{(-4)^2 + (3)^2 + (5)^2} = \sqrt{50} = 5\sqrt{2}$

$\cos\alpha = \dfrac{u_1}{\|\vec{u}\|}$

$\cos\alpha = \dfrac{-4}{5\sqrt{2}}$

$\alpha = \cos^{-1}\left(-\dfrac{4}{5\sqrt{2}}\right)$

$\alpha \approx 2.1721$

$\cos\beta = \dfrac{u_2}{\|\vec{u}\|}$

$\cos\beta = \dfrac{3}{5\sqrt{2}}$

$\beta \approx 1.1326$

$\cos\gamma = \dfrac{u_3}{\|\vec{u}\|}$

$\cos\gamma = \dfrac{5}{5\sqrt{2}}$

$\cos\gamma = \dfrac{1}{\sqrt{2}}$

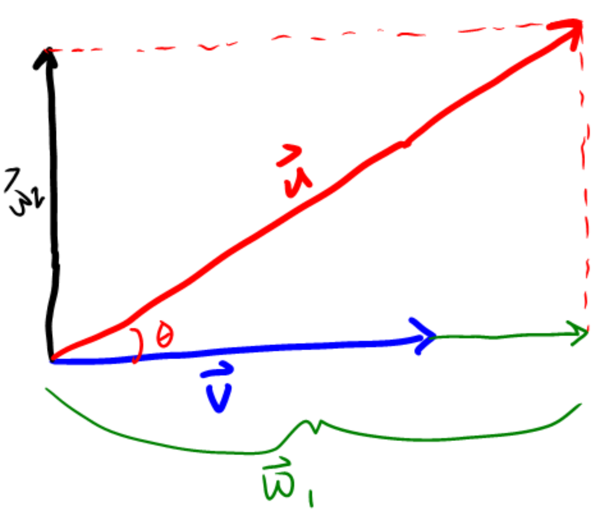
$\gamma = \dfrac{\pi}{4}$

DEFINITION OF PROJECTION AND VECTOR COMPONENTS

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors and let $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to $\mathbf{v}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$.

1. $\mathbf{w}_1$ is called the projection of $\mathbf{u}$ onto $\mathbf{v}$ or the vector component of $\mathbf{u}$ along $\mathbf{v}$, and is denoted by $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$.
2. $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$ is called the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$.

θ is acute



$\vec{w}_1 = \text{proj}_{\vec{v}}\,\vec{u}$ → projection of $\vec{u}$ onto $\vec{v}$

→ vector component of $\vec{u}$ along $\vec{v}$

$\vec{w}_2$ = vector component of $\vec{u}$ orthogonal to $\vec{v}$

THEOREM: PROJECTION USING THE DOT PRODUCT

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, then the projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by

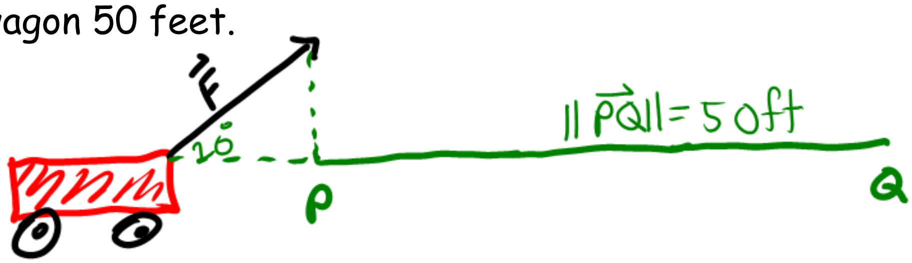
$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

DEFINITION OF WORK

The work W done by a constant force $\mathbf{F}$ as its point of application moves along the vector $\overrightarrow{PQ}$ is given by either of the following:

1. $W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\|\overrightarrow{PQ}\|$
2. $W = \mathbf{F}\cdot\overrightarrow{PQ}$

Example 5: A toy wagon is pulled by exerting a force of 25 pounds on a handle that makes a 20° angle with the horizontal. Find the work done in pulling the wagon 50 feet.



$\overrightarrow{PQ} = 50\hat{i}$

$\vec{F} = 25(\cos 20°\,\hat{i} + \sin 20°\,\hat{j})$

$W = \vec{F}\cdot\overrightarrow{PQ}$

$W = 25(\cos 20°\,\hat{i} + \sin 20°\,\hat{j})\cdot 50\hat{i}$

$W = 1250\cos 20° + 0$

$\boxed{W \doteq 1174.6 \text{ ft-lbs}}$