6. (4 points) Find the 4[th] Taylor polynomial for $f(x) = \dfrac{1}{x^2}$, centered at 2.

$f(x) = x^{-2}$

$f'(x) = -2x^{-3}$

$f''(x) = 6x^{-4}$

$f^{(3)}(x) = -24x^{-5}$

$f^{(4)}(x) = 120x^{-6}$

$f(2) = \frac{1}{4}$

$f'(2) = -\frac{2}{8} = -\frac{1}{4}$

$f''(2) = \frac{6}{16} = \frac{3}{8}$

$f^{(3)}(2) = \frac{-24}{32} = -\frac{3}{4}$

$f^{(4)}(2) = \frac{120}{64} = \frac{15}{8}$

$P_4(x) = \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3}{8}\cdot\frac{(x-2)^2}{2!} - \frac{3}{4}\cdot\frac{(x-2)^3}{3!} + \frac{\cancel{15}\,5}{8}\frac{(x-2)^4}{4!}$ $\quad 4\cdot\cancel{3}\cdot 2\cdot 1$

$$\boxed{P_4(x) = \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3}{16}(x-2)^2 - \frac{1}{8}(x-2)^3 + \frac{5}{64}(x-2)^4}$$

7. (8 points) Consider the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!(x-4)^n}{3^n}$.

a. State where the power series is centered.

$\boxed{c = 4}$

b. Find the **radius of convergence**.

$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)!(x-4)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n!(x-4)^n} \right| < 1 \rightarrow \lim_{n\to\infty} \left| \frac{(-1)(n+1)\cancel{n!}(x-4)}{3\cancel{n!}} \right| < 1$

$\lim_{n\to\infty} \left| \frac{(n+1)(x-4)}{3} \right| < 1$

converges only if $x = 4$ so $\boxed{R = 0}$

c. Find the **interval of convergence**. Be sure to check for convergence at the endpoints of the interval.

$\boxed{\{4\}\text{, no endpoints to check!}}$

c. $\sum_{n=1}^{\infty} n\left(\frac{10}{9}\right)^n$

Step 1: Identify test and its conditions (if applicable)

nth term test for divergence

Step 2: Run the test

$\lim_{n\to\infty} n\left(\frac{10}{9}\right)^n = \infty \neq 0$

Step 3: Conclusion

$\boxed{\sum_{n=1}^{\infty} n\left(\frac{10}{9}\right)^n \text{ diverges by the nth term test for divergence}}$

d. $\sum_{n=1}^{\infty}\left(2+3\sqrt[n]{n}\right)^n$

Step 1: Identify test and its conditions (if applicable)

Root test. $\sum_{n=1}^{\infty}(2+3\sqrt[n]{n})^n$ is a series.

Step 2: Run the test

$\lim_{n\to\infty} \sqrt[n]{\left|(2+3\sqrt[n]{n})^n\right|} = \lim_{n\to\infty}\left|(2+3\sqrt[n]{n})\right|$

$= \lim_{n\to\infty}\left|(2+3n^{1/n})\right|$

$= \lim_{n\to\infty} 2 + 3\lim_{n\to\infty} n^{1/n}$

$= 2+3(1)$

$= 5 > 1$

$y = n^{1/n}$

$\ln y = \ln n^{1/n}$

$\ln y = \frac{1}{n}\ln n$

$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1}$

D.S. $\frac{\infty}{\infty}$

$= 0$

$\ln y = 0$

$e^0 = y$

$1 = y$

Step 3: Conclusion

$\sum_{n=1}^{\infty}(2+3\sqrt[n]{n})^n$ diverges by the root test

e. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+4}$

Step 1: Identify test and its conditions (if applicable)

AST. $a_n = \frac{n^2}{n^2+4}$

Step 2: Run the test

$\lim_{n\to\infty} \frac{n^2/n^2}{\frac{n^2}{n^2}+\frac{4}{n^2}} = \lim_{n\to\infty} \frac{1}{1+4/n^2}$

$= \frac{1}{1+0}$

$= 1 \neq 0$

Step 3: Conclusion

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+4}$ diverges by the nth term test for divergence

$0.\overline{01} = 0.01010101\ldots$

4. (4 points) Consider the repeating decimal $0.\overline{01}$.

a. Write the repeating decimal as a geometric series.

$0.\overline{01} = 0.01 + 0.0001 + 0.000001 + \ldots$

$= 0.01(1 + 0.01 + 0.0001 + \ldots)$

$= \boxed{\sum_{n=0}^{\infty} 0.01(0.01)^n}$

b. Write its sum as the ratio of two integers.

$|r| = 0.01 < 1$

$S = \frac{0.01}{1-0.01} = \frac{0.01}{0.99} = \boxed{\frac{1}{99}}$

5. (40 points, 8 points each) Determine the convergence or divergence of the series.

a. $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$

and 2

Step 1: Identify test and its conditions (if applicable)

DCT. ~~Pet~~ series: $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ which is a divergent geometric series $\left[|r| = \frac{3}{2} \geq 1\right]$.

$a_n = \frac{3^n}{2^n - 1}$, $b_n = \left(\frac{3}{2}\right)^n$

Step 2: Run the test

$a_1 = 3 \geq b_1 = \frac{3}{2}$

$a_2 = 3 \geq b_2 = \frac{9}{4}$

$a_3 = \frac{27}{7} \geq b_3 = \frac{27}{8}$

$a_n, b_n > 0$ for all n.

Step 3: Conclusion

Since $\frac{3^n}{2^n - 1} \geq \left(\frac{3}{2}\right)^n$ for all n and $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ diverges, $\sum_{n=1}^{\infty} \frac{3^n}{2^n-1}$ also diverges by the DCT.

b. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Step 1: Identify test and its conditions (if applicable) Ratio Test.

$\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is a series with nonzero terms

Step 2: Run the test

$\lim_{n\to\infty} \left| \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(n+1)}\cancel{n!}\, n^n}{(n+1)^n \cancel{(n+1)}\, \cancel{n!}} \right|$

$= \lim_{n\to\infty} \left| \left(\frac{n}{n+1}\right)^n \right|$

* $= \frac{1}{e} < 1$

* let $y = \left(\frac{n}{n+1}\right)^n$

$\ln y = n \ln\left(\frac{n}{n+1}\right)$

$= \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}}$

$\lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}}$ D.S. $\frac{0}{0}$

$= \lim_{n\to\infty} \frac{\frac{(n+1)-n}{(n+1)^2} \Big/ \frac{n}{n+1}}{-\frac{1}{n^2}}$

$= \lim_{n\to\infty} \frac{\frac{1}{n(n+1)}}{-\frac{1}{n^2}}$

$= \lim_{n\to\infty} -\frac{n^{\cancel{2}}}{\cancel{n}(n+1)}$

$= -1$

$\ln y = -1 \leftrightarrow e^{-1} = y$

Step 3: Conclusion

$\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges absolutely by the ratio test

MATH 251/GRACEY
EXAM 4/CHAPTER 9.1-9.10
100 POINTS POSSIBLE

NAME Key

SHOW ALL WORK, GIVE EXACT ANSWERS (UNLESS OTHERWISE INDICATED) AND SUPPORT ALL RESULTS TO EARN FULL CREDIT

1. (6 points) A deposit of \$100 is made at the beginning of each month in an account at an annual interest rate of 3% compounded monthly. The balance in the account after n months is $A_n = 100(401)(1.0025^n - 1)$. Round to the nearest hundredth.

a. (2 POINTS) Compute the first four terms of the sequence $\{A_n\}$.

$\{100.25, 200.75, 301.50, 402.51\}$

b. (2 POINTS) Is $\{A_n\}$ a convergent sequence? Explain.

$\lim_{n\to\infty} (100)(401)(1.0025^n - 1) = \infty$

$\boxed{\{A_n\} \text{ is not a convergent sequence because the limit above is not finite.}}$

c. (2 POINTS) Find the balance in the account after 1 year.

$A_{12} = (100)(401)(1.0025^{12} - 1) = \boxed{1219.68}$

2. (3 points) Determine the convergence or divergence of the **sequence** with the given nth term. If the sequence converges, find its limit.

$$a_n = \frac{\sqrt{n}}{\ln n}$$

$\lim_{n\to\infty} \frac{\sqrt{n}}{\ln n} = \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} = \infty \Rightarrow \boxed{a_n = \frac{\sqrt{n}}{\ln n} \text{ diverges}}$

D.S. 8/18

3. (5 points) Find the sum of the convergent telescoping series.

$\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right) + \cdots$

$= \frac{1}{2} - \lim_{n\to\infty} \left(\frac{1}{(n+1)+1} - \frac{1}{(n+1)+2} \right)$

$= \frac{1}{2} - 0$

$= \boxed{\frac{1}{2}}$