MATH 251/GRACEY
EXAM 5/CHAPTER 10.1-10.5
SHOW ALL WORK FOR FULL CREDIT/GIVE EXACT ANSWERS UNLESS OTHERWISE INDICATED

NAME Key

1. (18 POINTS) Consider the conic section $y^2 - 6y - 3x^2 + 12x - 6 = 0$.

a. (2 POINTS) Write the equation in standard form. You may round to the nearest tenth.

$$\left(y^2-6y+(-3)^2\right)-3\left(x^2-4x+(-2)^2\right)=6-12+9$$
$$(y-3)^2-3(x-2)^2=3$$
$$\frac{(y-3)^2}{(\sqrt{3})^2}-\frac{(x-2)^2}{(1)^2}=1$$

hyperbola

b. (8 POINTS) Analyze the equation. If the equation does not have one of the following characteristics, write "none".

i. Find the vertex or the center.

$(2,3)$

ii. Find the focus or foci.

$c^2=a^2+b^2$

$c=\sqrt{3+1}$

$c=2$

$F_1: (2,3+2)$; $F_2: (2,3-2)$

$F_1: (2,5)$; $F_2: (2,1)$

iii. Write the equation for the directrix.

NONE

iv. Write the equation for the asymptotes.

$y=\frac{\sqrt{3}}{1}(x-2)+3$ or $y=-\frac{\sqrt{3}}{1}(x-2)+3$

$\boxed{y=\sqrt{3}(x-2)+3 \text{ or } y=-\sqrt{3}(x-2)+3}$

c. (6 POINTS) Find the zeros of the derivative.

$$\frac{d}{dx}\left(y^2-6y-3x^2+12x-6\right)=\frac{d}{dx}(0)$$
$$2yy'-6y'-6x+12=0$$
$$yy'-3y'-3x+6=0$$
$$y'(y-3)=3x-6$$
$$y'=\frac{3(x-2)}{y-3}$$

$0=3(x-2)$

$0=x-2$

$x=2$

so...

$\frac{(y-3)^2}{3}-\frac{(2-2)^2}{1}=1$

$(y-3)^2=3$

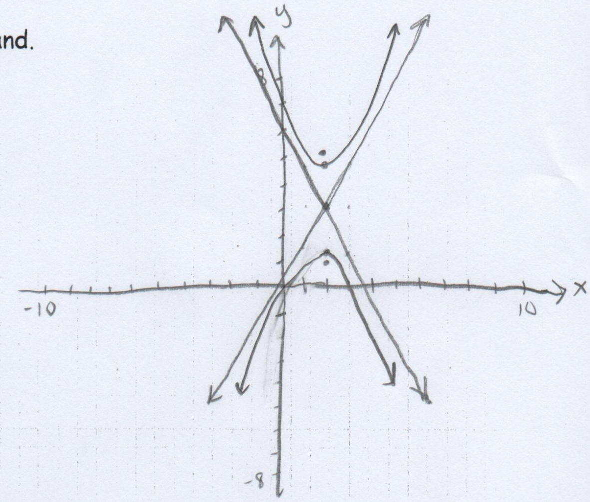
$y-3=\pm\sqrt{3}$

$\rightarrow y=3\pm\sqrt{3}$

$\boxed{\text{The derivative is zero at } (2,3-\sqrt{3}) \text{ and at } (2,3+\sqrt{3})}$

d. (2 POINTS) Sketch the graph by hand.

$$\frac{(y-3)^2}{(\sqrt{3})^2} - \frac{(x-2)^2}{(1)^2} = 1$$



$a = \sqrt{3} \approx 1.7$ | center: $(2,3)$

$b = 1$

$c^2 = a^2 + b^2$

$c = \sqrt{3+1}$

$c = 2$

2. (5 POINTS) Find the equation of the ellipse with

center: $(0,0)$ and solution points: $(1,2)$ and $(2,0)$.

$h=0,\ k=0$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at $(1,2)$: $\frac{1^2}{a^2} + \frac{2^2}{b^2} = 1$

$a^2b^2\left(\frac{1}{a^2} + \frac{4}{b^2}\right) = (1)a^2b^2$

$b^2 + 4a^2 = a^2b^2$

$4a^2 + b^2 = a^2b^2$

at $(2,0)$: $\frac{2^2}{a^2} + \frac{0^2}{b^2} = 1$

$\frac{4}{a^2} = 1$

$a^2 = 4$

So...

$4(4) + b^2 = (4)b^2$

$16 + b^2 = 4b^2$

$16 = 3b^2$

$\frac{16}{3} = b^2$

$$\frac{x^2}{4} + \frac{y^2}{\left(\frac{16}{3}\right)} = 1$$

$$\boxed{\frac{x^2}{4} + \frac{3y^2}{16} = 1}$$

3. (5 POINTS) Classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

a. $1 - x + 2y + 2y^2 = 0$ $\boxed{\text{parabola}}$

b. $2x - x^2 + y + y^2 - 1 = 9$ $\boxed{\text{hyperbola}}$

c. $-x^2 - y^2 = -12$ $\boxed{\text{circle}}$

d. $-x^2 - x + y^2 - 5y = 1 + x^2$ $\boxed{\text{hyperbola}}$

e. $2x^2 - 8x + y^2 + 5y = 6$ $\boxed{\text{ellipse}}$

4. (16 POINTS) Consider the parametric equation $2x = \sin\theta$ and $y - 1 = \cos\theta$.

$x = \frac{\sin\theta}{2}$ $\quad$ $y = \cos\theta + 1$

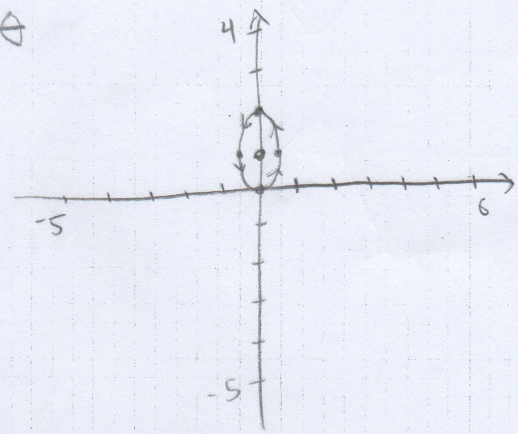
a. (10 POINTS) Eliminate the parameter and graph the parametric equation by hand, indicating the orientation.

$4x^2 = \sin^2\theta, \ (y-1)^2 = \cos^2\theta$

$\sin^2\theta + \cos^2\theta = 1$

$4x^2 + (y-1)^2 = 1$

$\frac{x^2}{\left(\frac{1}{2}\right)^2} + \frac{(y-1)^2}{(1)^2} = 1$



b. (6 POINTS) Evaluate $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-\sin\theta}{\left(\frac{\cos\theta}{2}\right)} = -2\left(\frac{2x}{y-1}\right) = \boxed{-\frac{4x}{y-1}}$ or $\boxed{-2\tan\theta}$

5. (12 POINTS) Consider the polar equation $r = 4\cos\theta$.

a. Find all points of horizontal tangency to the curve.

$y = r\sin\theta$

$\frac{dy}{d\theta} = r\cos\theta + r'\sin\theta$

$x = r\cos\theta$

$\frac{dx}{d\theta} = -r\sin\theta + r'\cos\theta$

$r' = -4\sin\theta$

$\frac{dy}{dx} = \frac{(4\cos\theta)(\cos\theta) - (4\sin\theta)(\sin\theta)}{(-4\cos\theta)(\sin\theta) - 4\sin\theta\cos\theta}$

$\frac{dy}{dx} = \frac{4\cos^2\theta - 4\sin^2\theta}{-8\cos\theta\sin\theta}$

$0 = 4\cos^2\theta - 4\sin^2\theta$

$0 = 4\cos 2\theta$

$0 = \cos 2\theta$

$2\theta = -\frac{\pi}{2}, \frac{\pi}{2}$

$\theta = -\frac{\pi}{4}, \frac{\pi}{4}$

points of Horizontal tangency at: $(2\sqrt{2}, -\frac{\pi}{4}), (2\sqrt{2}, \pi/4)$

b. Find all points of vertical tangency to the curve

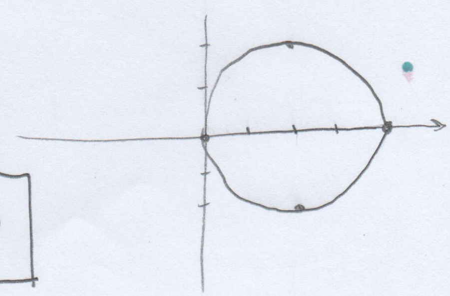
$0 = -8\cos\theta\sin\theta$

$\cos\theta = 0$ or $\sin\theta = 0$

$\theta = -\frac{\pi}{2}, \frac{\pi}{2}$ or $\theta = 0$

yield same point

points of vertical tangency at: $(4, 0), (0, \pi/2)$



6. (10 POINTS) Find the arc length of the curve

$x = \arcsin t$ and $y = \ln\sqrt{1-t^2}$ on the interval $0 \le t \le \frac{1}{2}$.

$\frac{dx}{dt} = \frac{1}{\sqrt{1-t^2}}$ $\quad \frac{dy}{dt} = \frac{1}{2}\left(\frac{-2t}{1-t^2}\right)$

$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\frac{1}{1-t^2} + \frac{t^2}{(1-t^2)^2}}$

$= \sqrt{\frac{1-t^2+t^2}{(1-t^2)^2}}$

$= \sqrt{\frac{1}{(1-t^2)^2}}$

$= \frac{1}{1-t^2}$

$s = \int_0^{1/2} \frac{dt}{1-t^2}$

$s = \frac{1}{2} \ln\left|\frac{1+t}{1-t}\right| \Big|_0^{1/2}$

$s = \frac{1}{2}\left[\ln\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right) - \ln\left(\frac{1}{1}\right)\right]$

$\boxed{s = \frac{1}{2}\ln 3 \text{ units} \quad \text{or} \quad s = \ln\sqrt{3} \text{ units}}$

7. (4 POINTS) Find two sets of polar coordinates for the rectangular coordinate

$\left(-3, -\sqrt{3}\right)$. QIII

$r = \sqrt{x^2+y^2}$

$r = \sqrt{(-3)^2 + (-\sqrt{3})^2}$

$r = \sqrt{9+3}$

$r = \sqrt{12}$

$r = 2\sqrt{3}$

$\tan\theta = \frac{-\sqrt{3}}{-3}$

$\tan\theta = \frac{-1/2}{-\sqrt{3}/2}$

$\theta = \frac{7\pi}{6}$

$\boxed{(2\sqrt{3}, 7\pi/6) \text{ and } (-2\sqrt{3}, \pi/6)}$

8. (30 POINTS) Consider the polar equation $r = 5\sin 3\theta$.

a. (4 POINTS) Sketch a graph of the polar equation by hand.

rose graph with 3 petals

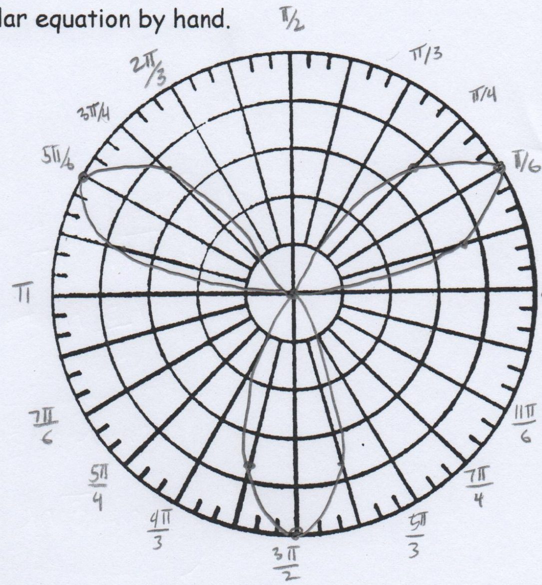
$2\pi \div 3 = \frac{2\pi}{3}$

1st petal's point is at $\frac{\pi}{6}$ so

$\frac{\pi}{6} + \frac{2\pi}{3} = \frac{5\pi}{6}$

$\frac{5\pi}{6} + \frac{2\pi}{3} = \frac{9\pi}{6} = \frac{3\pi}{2}$

r	θ
0	0
5	$\pi/6$
3.7	$\pi/4$
3.7	$\pi/12$



b. (6 POINTS) Find the tangents at the pole.

$0 = 5\sin 3\theta$

$0 = \sin 3\theta$

$3\theta = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$

$\theta = 0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3$

$\theta = 0$ and $\theta = \pi$ same radial line, $\theta = \pi/3$ and $\theta = 4\pi/3$ same radial line, $\theta = 2\pi/3$ and $\theta = 5\pi/3$ same radial line

$r' = 15\cos 3\theta$

at $\theta = 0$: $r' = 15 \neq 0$ ✓

at $\theta = \pi/3$: $r' = -15 \neq 0$ ✓

at $\theta = 2\pi/3$: $r' = 15 \neq 0$ ✓

tangents at the pole: $\boxed{\theta = 0,\ \theta = \pi/3,\ \theta = \frac{2\pi}{3}}$

c. (6 POINTS) Find the area of the interior.

$$A = 3\left[\frac{1}{2}\int_0^{\pi/3} (5\sin 3\theta)^2 \, d\theta\right]$$

$$A = \frac{3}{2}\int_0^{\pi/3} 25\sin^2 3\theta \, d\theta$$

$$A = \frac{75}{2}\int_0^{\pi/3} \frac{1 - \cos 6\theta}{2} \, d\theta$$

$$A = \frac{75}{4}\left(\theta - \frac{\sin 6\theta}{6}\right)\Big|_0^{\pi/3}$$

$$A = \frac{75}{4}\left[\left(\frac{\pi}{3} - 0\right) - (0 - 0)\right]$$

$$\boxed{A = \frac{25\pi}{4} \text{ sq. units}}$$