MATH 251/GRACEY NAME Key

EXAM 3/CHAPTER 8

100 POINTS POSSIBLE/SHOW ALL WORK FOR FULL CREDIT/EXACT ANSWERS ONLY UNLESS OTHERWISE INDICATED/CALCULATOR OK

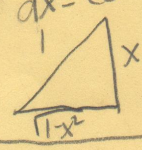
NO TABLE FORMULAS FROM SECTION 8.6 MAY BE USED FOR ANY REASON!

1. (48 POINTS) Find FOUR out of the following FIVE the integrals.

a. $\int x\arcsin x\,dx = \frac{x^2\arcsin x}{2} - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

$u = \arcsin x$, $du = \frac{dx}{\sqrt{1-x^2}}$, $dv = x\,dx$, $v = \frac{x^2}{2}$

$x = \sin\theta$, $\sqrt{1-x^2} = \cos\theta$, $x^2 = \sin^2\theta$, $dx = \cos\theta\,d\theta$



$$= \frac{x^2\arcsin x}{2} - \frac{1}{2}\int \frac{\sin^2\theta}{\cancel{\cos\theta}}(\cancel{\cos\theta}\,d\theta)$$

$$= \frac{x^2\arcsin x}{2} - \frac{1}{4}\int (1-\cos 2\theta)\,d\theta$$

$$= \frac{x^2\arcsin x}{2} - \frac{\theta}{4} + \frac{\sin 2\theta}{8} + C$$

$\frac{2\sin\theta\cos\theta}{\cancel{4}\,8}$

$$= \boxed{\frac{x^2\arcsin x}{2} - \frac{\arcsin x}{4} + \frac{x\sqrt{1-x^2}}{4} + C}$$

b. $\int \ln\sqrt{x^2-1}\,dx$

$u = \ln(x^2-1)$, $du = \frac{2x}{x^2-1}dx$, $dv = dx$, $v = x$

$$= \frac{1}{2}\int \ln(x^2-1)\,dx$$

$$= \frac{1}{2}\left[x\ln(x^2-1) - 2\int \frac{x^2}{x^2-1}\,dx\right]$$

$$= \frac{1}{2}x\ln(x^2-1) - \int\left(1 + \frac{1}{x^2-1}\right)dx$$

$$= \frac{1}{2}x\ln(x^2-1) - x - \int\left(-\frac{1}{2(x+1)} + \frac{1}{2(x-1)}\right)dx$$

$$= \frac{x}{2}\ln(x^2-1) - x + \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + C$$

$$= \frac{x\ln(x^2-1)}{2} - x + \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C$$

$$= \boxed{x\ln\sqrt{x^2-1} - x + \ln\sqrt{\left|\frac{x+1}{x-1}\right|} + C}$$

$$\begin{array}{r} 1 + \frac{1}{x^2-1} \\ (x^2-1)\overline{)\,x^2+0x+0} \\ -(x^2\quad -1) \\ \hline 1 \end{array}$$

PFD:

$\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$

$0x + 1 = Ax - A + Bx + B$

$0x + 1 = (A+B)x + (-A+B)$

$A + B = 0$

$-A + B = 1$

$2B = 1$

$B = \frac{1}{2}$

$A = -\frac{1}{2}$

c. $\int \sqrt{25-16x^2}\,dx = \int (5\cos\theta)\left(\frac{5}{4}\cos\theta\, d\theta\right)$

$4x = 5\sin\theta$

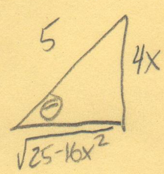
$\sqrt{25-16x^2} = \sqrt{25-(5\sin\theta)^2}$

$= \sqrt{25(1-\sin^2\theta)}$

$= 5\cos\theta$

$4\,dx = 5\cos\theta\, d\theta$

$dx = \frac{5}{4}\cos\theta\, d\theta$



$\sin\theta = \frac{4x}{5}$

$\theta = \arcsin\frac{4x}{5}$

$= \frac{25}{4}\int \cos^2\theta\, d\theta$

$= \frac{25}{8}\int (1+\cos 2\theta)\, d\theta$

$= \frac{25}{8}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$

$= \frac{25}{8}\left(\arcsin\left(\frac{4x}{5}\right) + \left(\frac{4x}{5}\right)\left(\frac{\sqrt{25-16x^2}}{5}\right)\right) + C$

$= \boxed{\frac{25}{8}\arcsin\left(\frac{4x}{5}\right) + \frac{x\sqrt{25-16x^2}}{2} + C}$

$\frac{\sin 2\theta}{2} = \sin\theta\cos\theta$

d. $\int \sin^4 3\theta\, d\theta = \int (\sin^2 3\theta)^2\, d\theta$

$= \int \left(\frac{1-\cos 6\theta}{2}\right)^2 d\theta$

$= \frac{1}{4}\int (1 - 2\cos 6\theta + \cos^2 6\theta)\, d\theta$

$= \frac{1}{4}\left(\theta - \frac{2\sin 6\theta}{6} + \frac{1}{2}\int (1+\cos 12\theta)\, d\theta\right)$

$= \frac{\theta}{4} - \frac{\sin 6\theta}{12} + \frac{\theta}{8} + \frac{\sin 12\theta}{96} + C$

$= \boxed{\frac{3\theta}{8} - \frac{\sin 6\theta}{12} + \frac{\sin 12\theta}{96} + C}$

e. $\int \frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x} dx = \int (2x-3)dx + \int \frac{x-4}{x^2-x} dx$

$$\begin{array}{r} 2x-3+\frac{x-4}{x^2-x} \\ (x^2-x)\overline{)2x^3-5x^2+4x-4} \\ -(2x^3-2x^2) \\ \hline -3x^2+4x \\ -(-3x^2+3x) \\ \hline x-4 \end{array}$$

$\frac{x-4}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$

$x-4 = Ax - A + Bx$

$x-4 = (A+B)x + (-A)$

$A+B=1$

$-A=-4$

$A=4$

$B=-3$

$= x^2 - 3x + \int \left(\frac{4}{x} - \frac{3}{x-1}\right) dx$

$= x^2 - 3x + 4\ln|x| - 3\ln|x-1| + C$

$= x^2 - 3x + \ln x^4 - \ln(|x-1|)^3 + C$

$= \boxed{x^2 - 3x + \ln \frac{x^4}{(|x-1|)^3} + C}$

2. (45 POINTS) Evaluate THREE out of the following FOUR definite integrals. If the integral is improper, determine whether the improper integral diverges or converges. Evaluate the integral if it converges.

proper

a. $\int_0^{\pi/3} \tan^2 x dx = \int_0^{\pi/3} (\sec^2 x - 1) dx$

$= \tan x - x \Big|_0^{\pi/3}$

$= \left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - (\tan 0 - 0)$

$= \sqrt{3} - \frac{\pi}{3} - 0$

$= \boxed{\frac{1}{3}\left(3\sqrt{3} - \pi\right)}$

proper

b. $\int_0^2 e^x \cos 2x\,dx = e^x\cos 2x + 2\int e^x \sin 2x\,dx$

$u=\cos 2x$
$du=-2\sin 2x\,dx$
$dv=e^x dx$
$v=e^x$

$u=\sin 2x$
$du=2\cos 2x\,dx$
$dv=e^x dx$
$v=e^x$

$\int_0^2 e^x\cos 2x\,dx = e^x\cos 2x + 2\left[e^x\sin 2x - 2\int e^x\cos 2x\,dx\right]$

$\int_0^2 e^x\cos 2x\,dx = e^x\cos 2x + 2e^x\sin 2x - 4\int e^x\cos 2x\,dx$

$5\int_0^2 e^x\cos 2x\,dx = e^x\cos 2x + 2e^x\sin 2x$

$\int_0^2 e^x\cos 2x\,dx = \left[\frac{e^x\cos 2x}{5}(1+2\sin 2x)\right]_0^2$

$\int_0^2 e^x\cos 2x\,dx = \frac{e^2\cos 4}{5}(1+2\sin 4) - \frac{e^0\cos 0}{5}(1+2\sin 0)$

$\boxed{\int_0^2 e^x\cos 2x\,dx = \frac{1}{5}\left(e^2\cos 4(1+2\sin 4)-1\right)}$

improper

c. $\int_0^e \ln x\,dx$

$u=\ln x \quad dv=dx$
$du=\frac{dx}{x} \quad v=x$

$=\lim_{a\to 0^+}\int_a^e \ln x\,dx$

$=\lim_{a\to 0^+}\left[x\ln x - \int_a^e dx\right]$

$=\lim_{a\to 0^+}\left[x\ln x - x\right]_a^e$

$=\lim_{a\to 0^+}\left[(e\ln e - e) - a\ln a - a\right]$

$= 0 - \lim_{a\to 0^+}\left(\frac{\ln a}{\frac{1}{a}} - a\right)$

$= -\lim_{a\to 0^+}\frac{\frac{1}{a}}{-\frac{1}{a^2}} - 0$

$= -\lim_{a\to 0^+} -a$

$= \boxed{0} \rightarrow$ $\boxed{\int_0^e \ln x\,dx \text{ converges to zero}}$

improper

d. $\int_1^\infty \frac{7}{\sqrt[4]{x^5}}dx$

$=7\int_1^\infty \frac{1}{x^{5/4}}dx$

$p=\frac{5}{4}>1$

$=7\left(\frac{1}{\frac{5}{4}-1}\right)$

$=\boxed{28}$

3. (7 POINTS) For the following limit:

$$\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right)$$

a. Describe the type of indeterminate form (if any) that is obtained by direct substitution.

$\infty \cdot 0$

b. Evaluate the limit, using L'Hôpital's Rule if necessary.

D.S. $\frac{0}{0}$

$$\lim_{x\to\infty} \frac{\tan(\frac{1}{x})}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\sec^2(\frac{1}{x})\cdot(-\cancel{\frac{1}{x^2}})}{-\cancel{\frac{1}{x^2}}}$$

$$= \sec^2 0$$

$$= (1)^2$$

$$= \boxed{1}$$