MATH 251/GRACEY
WORKSHEET/9.3-9.6

NAME Key

DETERMINE THE CONVERGENCE OR DIVERGENCE OF THE SERIES.

1. $$\sum_{n=1}^{\infty} \frac{5}{n+\sqrt{n^2+4}}$$

① Try n-th term test:

$$\lim_{n\to\infty} \frac{5}{n+\sqrt{n^2+4}} = 0$$

crap we need to test differently...

② Try D.C.T.

choose $a_n = \frac{5}{4n}$, $b_n = \frac{5}{n+\sqrt{n^2+4}}$

$a_1 = 1.25 < b_1 \approx 1.55$

$a_2 = .625 < b_2 \approx 1.03$

So we can use the DCT

③ $\frac{5}{4}\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series [p=1]

Therefore $\sum_{n=1}^{\infty} \frac{5}{n+\sqrt{n^2+4}}$ also diverges by the D.C.T.

2. $$\sum_{n=0}^{\infty} \frac{n^{k-1}}{n^k+1},\ k>2 = \sum_{n=0}^{\infty} \frac{n^k}{n(n^k+1)}$$

b_n

$$\frac{n^{k-1}}{n^k} = \frac{1}{n}$$

$$a_n = \frac{n^k}{n(n^k+1)}$$

conditions:
(using the L.C.T.)

✓ $a_n > 0$, $b_n > 0$ for all n

test

$$\frac{a_n}{b_n} = \frac{\frac{n^k}{\cancel{n}(n^k+1)}}{1/\cancel{n}} = \frac{n^k}{n^k+1}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^k}{n^k+1} = 1$$

Conclusion

$\sum_{n=0}^{\infty} \frac{n^{k-1}}{n^k+1}, k>2$ diverges since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series

3. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

③ $\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$ is divergent by the ratio test

Ratio Test

① all positive so we can use the test

② $\frac{a_{n+1}}{a_n} = \frac{(2(n+1))!}{(n+1)^5} \cdot \frac{n^5}{(2n)!} = \frac{(2n+2)(2n+1)\cancel{(2n)}!\, n^5}{\cancel{(2n)}!\,(n+1)^5}$

$(2n+2)!$

degree of 7

degree of 5

$\lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)n^5}{(n+1)^5} \right| = \infty$

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4. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ A.S.T.

① $a_n = \frac{1}{n!}$

$a_0 = \frac{1}{0!} = \frac{1}{1} = 1$

$a_1 = \frac{1}{1!} = 1$

$a_2 = \frac{1}{2!} = \frac{1}{2}$

so $0 < a_{n+1} \le a_n$ ✓

② $\lim_{n\to\infty} \frac{1}{n!} = 0$ ✓

conclusion: $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ converges by the alternating series test.

5. Use the Integral Test to determine the convergence or divergence of the series

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$$

Conditions

$f(n) = a_n$ for all n.

continuous on $[2, \infty)$

$$f'(x) = \frac{\frac{1}{x}(x^3) - (\ln \quad)3x^2}{x^6} = \frac{\cancel{x}^{2}(1 - 3\ln x)}{x^{\cancel{6}4}}$$

$$1 - 3\ln x < 0$$
$$\ln x > \tfrac{1}{3} \longrightarrow \begin{array}{l}\ln 1 = 0 \\ \ln 2 \approx .693 > \tfrac{1}{3}\end{array}$$

so from $N=2$ f is decreasing.

Test

$$\int_2^{\infty} \frac{\ln x}{x^3}\,dx = \lim_{b\to\infty}\left[\frac{\ln x}{-2x^2} + \int_2^b \frac{dx}{x \cdot 2x^2}\right]$$

$u = \ln x$
$du = \frac{dx}{x}$
$dv = x^{-3}dx$
$v = -\frac{1}{2x^2}$

$$= \lim_{b\to\infty}\left[-\frac{\ln x}{2x^2} + \frac{1}{2}\int_2^b x^{-3}\,dx\right]$$

$$= \lim_{b\to\infty}\left[-\frac{\ln x}{2x^2} - \frac{1}{4x^2}\right]_2^b$$

$$= \left[\lim_{b\to\infty}\left(-\frac{\ln b}{2b^2}\right) - \lim_{b\to\infty}\frac{1}{4b^2}\right] - \left[-\frac{\ln 2}{2(2)^2} - \frac{1}{4(2)^2}\right]$$

L'Hôpital's Rule

$$= \lim_{b\to\infty}\left(-\frac{\frac{1}{b}}{4b}\right) - 0 + \frac{\ln 2}{8} + \frac{1}{16}$$

$$= \lim_{b\to\infty}\left(-\frac{1}{4b^2}\right) + \underline{1 + 2\ln 2}$$

$$= 0 + \frac{1 + \ln 4}{16} = \frac{1 + \ln 4}{16}$$

converges by the Integral Test