ATH 251/GRACEY
ORKSHEET/5.8

HYPERBOLIC FUNCTIONS, INVERSE HYPERBOLIC FUNCTIONS AND IDENTITIES

$\sinh x = \dfrac{e^x - e^{-x}}{2}$ $\cosh x = \dfrac{e^x + e^{-x}}{2}$ $\tanh x = \dfrac{\sinh x}{\cosh x}$, $\cosh x \neq 0$

$\operatorname{csch} x = \dfrac{1}{\sinh x}$, $\sinh x \neq 0$ $\operatorname{sech} x = \dfrac{1}{\cosh x}$, $\cosh x \neq 0$ $\coth x = \dfrac{\cosh x}{\sinh x}$, $\sinh x \neq 0$

$\cosh^2 x - \sinh^2 x = 1$ $\tanh^2 x + \operatorname{sech}^2 x = 1$ $\coth^2 x - \operatorname{csch}^2 x = 1$

$\sinh^2 x = \dfrac{-1 + \cosh 2x}{2}$ $\cosh^2 x = \dfrac{1 + \cosh 2x}{2}$

$\sinh 2x = 2 \sinh x \cosh x$ $\cosh 2x = \cosh^2 x + \sinh^2 x$

Function	Domain
$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$	$(-\infty, \infty)$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$	$[1, \infty)$
$\tanh^{-1} x = \dfrac{1}{2} \ln \dfrac{1+x}{1-x}$	$(-1, 1)$
$\coth^{-1} x = \dfrac{1}{2} \ln \dfrac{x+1}{x-1}$	$(-\infty, -1) \cup (1, \infty)$
$\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{\lvert x \rvert}\right)$	$(-\infty, 0) \cup (0, \infty)$
$\operatorname{sech}^{-1} x = \ln \dfrac{1 + \sqrt{1-x^2}}{x}$	$[0, 1)$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

1. Evaluate the expression without using a calculator.

a. $\sinh(-1)$

$$= \frac{e^{(-1)} - e^{-(-1)}}{2} = \frac{\frac{1}{e} - e}{2} = \boxed{\frac{1-e^2}{2e}}$$

b. $\tanh\left(\frac{1}{2}\right)$

$$\frac{e^{1/2} - e^{-1/2}}{e^{1/2} + e^{-1/2}} = \frac{\cancel{e^{-1/2}}(e-1)}{\cancel{e^{-1/2}}(e+1)} = \boxed{\frac{e-1}{e+1}}$$

c. $\operatorname{sech}(-3)$

$$\frac{2}{e^{-3} + e^{-(-3)}} = \frac{2}{e^{-3}(1+e^6)} = \boxed{\frac{2e^3}{1+e^6}}$$

d. $\coth^{-1}(1)$

$$\frac{1}{2}\ln\frac{(1)+1}{(1)-1} \longrightarrow \text{undefined}$$

e. $\cosh^{-1}(5)$

$$= \ln\left((5) + \sqrt{(5)^2 - 1}\right)$$

$$= \ln\left(5 + \sqrt{24}\right)$$

$$= \boxed{\ln\left(5 + 2\sqrt{6}\right)}$$

f. $\tanh^{-1}(0)$

$$= \frac{1}{2}\ln\left(\frac{1+(0)}{1-(0)}\right)$$

$$= \frac{1}{2}\ln(1)$$

$$= \frac{1}{2}(0)$$

$$= \boxed{0}$$

2. Verify the identity.

$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

RHS: $\cosh x \cosh y - \sinh x \sinh y = \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) - \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$

$$= \frac{1}{4}\left[\cancel{e^{x+y}} + e^{x-y} + e^{y-x} + \cancel{e^{-(x+y)}} - \cancel{e^{x+y}} + e^{x-y} + e^{y-x} - \cancel{e^{-(x+y)}}\right]$$

$$= \frac{1}{4}\left[2e^{x-y} + 2e^{y-x}\right]$$

$$= \frac{e^{x-y} + e^{-(x-y)}}{2}$$

$$= \cosh(x-y) \,//$$

DERIVATIVES AND INTEGRALS OF HYPERBOLIC FUNCTIONS AND INVERSE HYPERBOLIC FUNCTIONS

$$\frac{d}{dx}[\sinh u] = (\cosh u)u' \qquad \frac{d}{dx}[\cosh u] = (\sinh u)u'$$

$$\frac{d}{dx}[\tanh u] = (\operatorname{sech}^2 u)u' \qquad \frac{d}{dx}[\coth u] = -(\operatorname{csch}^2 u)u'$$

$$\frac{d}{dx}[\operatorname{sech} u] = -(\operatorname{sech} u \tanh u)u' \qquad \frac{d}{dx}[\operatorname{csch} u] = -(\operatorname{csch} u \coth u)u'$$

$$\frac{d}{dx}\left[\sinh^{-1} u\right] = \frac{u'}{\sqrt{u^2+1}} \qquad \frac{d}{dx}\left[\cosh^{-1} u\right] = \frac{u'}{\sqrt{u^2-1}}$$

$$\frac{d}{dx}\left[\tanh^{-1} u\right] = \frac{u'}{1-u^2} \qquad \frac{d}{dx}\left[\operatorname{sech}^{-1} u\right] = \frac{-u'}{u\sqrt{1-u^2}}$$

$$\frac{d}{dx}\left[\coth^{-1} u\right] = \frac{u'}{1-u^2} \qquad \frac{d}{dx}\left[\operatorname{csch}^{-1} u\right] = \frac{-u'}{|u|\sqrt{1+u^2}}$$

3. Find the derivative of the function. **Write your result as a single ~~trigonometric~~ expression**.

a. $h(x) = x\sinh\left(x^2\right)$

$$h'(x) = (1)\sinh(x^2) + x\left[(\cosh x^2)(2x)\right]$$
$$= \boxed{\sinh x^2 + 2x^2\cosh x^2}$$

b. $f(x) = x - \coth(x)$

$$f'(x) = 1 - (-\operatorname{csch}^2 x) = 1 + \operatorname{csch}^2 x = \boxed{\coth^2 x}$$

$u = \frac{t}{4}$

$u' = \frac{1}{4}$

c. $g(t) = \cosh^{-1}\left(\frac{t}{4}\right)$

$$g'(t) = \frac{\frac{1}{4}}{\sqrt{\left(\frac{t}{4}\right)^2 - 1}} = \frac{\frac{1}{4}}{\sqrt{\frac{t^2}{16} - 1}} = \frac{1}{4\sqrt{\frac{t^2-16}{16}}} = \frac{1}{\frac{\cancel{4}\sqrt{t^2-16}}{\cancel{4}}} = \boxed{\frac{1}{\sqrt{t^2-16}}}$$

d. $f(x) = \left(\operatorname{sech}^{-1}(-x)\right)^2$

$$f'(x) = 2\left(\operatorname{sech}^{-1}(-x)\right)^1\left(\frac{-(-1)}{(-x)\sqrt{1-(-x)^2}}\right)$$
$$= \boxed{-\frac{2\operatorname{sech}^{-1}(-x)}{x\sqrt{1-x^2}}}$$

c. $\int \frac{3}{\sqrt{x}\sqrt{9+x}}dx = 3\int \frac{2\cancel{\sqrt{x}}\,du}{\cancel{\sqrt{x}}\sqrt{3^2+(\sqrt{x})^2}}$

$a=3$
$u=\sqrt{x}$
$du=\frac{dx}{2\sqrt{x}}$
$2\sqrt{x}\,du=dx$

$= 6\int \frac{du}{\sqrt{3^2+u^2}}$

$= 6\int \frac{du}{\sqrt{u^2+3^2}}$

$= 6\ln\left(u+\sqrt{u^2+3^2}\right)+C$

$= \boxed{6\ln\left[\sqrt{x}+\sqrt{x+9}\right]+C}$

d. $\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}} = \int \frac{dx}{(x+2)\sqrt{(x+2)^2+2^2}}$

$(x^2+4x+4)+8-4$
$(x+2)^2+4$

$a=2$
$u=x+2$
$du=dx$

$= \int \frac{dx}{(x+2)\sqrt{2^2+(x+2)^2}}$

$= \int \frac{du}{u\sqrt{2^2+u^2}} = \boxed{-\frac{1}{2}\ln\left(\frac{2+\sqrt{4+(x+2)^2}}{|x+2|}\right)+C}$

e. $\int \frac{1}{1-4x-2x^2}dx = \int \frac{1/2}{\frac{3}{2}-\frac{2(x+1)^2}{2}}dx = \frac{1}{2}\int \frac{dx}{\frac{3}{2}-(x+1)^2}$

$-2(x^2+2x+1)+1+2$
$3-2(x+1)^2$

$a=\sqrt{\frac{3}{2}}$
$u=x+1$
$du=dx$

$= \frac{1}{2}\int \frac{du}{\left(\sqrt{\frac{3}{2}}\right)^2-u^2}$

$= \frac{1}{2\sqrt{\frac{3}{2}}}\cdot\frac{1}{2}\ln\left|\frac{\sqrt{\frac{3}{2}}+u}{\sqrt{\frac{3}{2}}-u}\right|+C$

$= \boxed{\frac{\sqrt{6}}{12}\ln\left|\frac{\frac{\sqrt{6}}{2}+(x+2)}{\frac{\sqrt{6}}{2}-(x+2)}\right|+C}$

$u=e^x+e^{-x}$
$du=(e^x-e^{-x})dx$

f. $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx$

$= \int \frac{du}{u}$

$= \ln|u|+C$

$= \boxed{\ln(e^x+e^{-x})+C}$