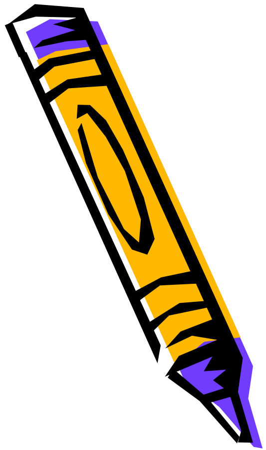
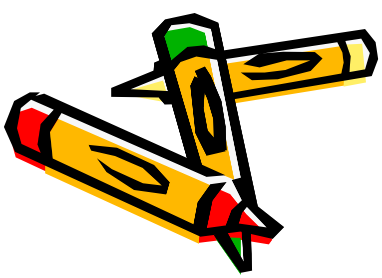
Chapter 9.6

THE RATIO AND ROOT TESTS

After you finish your HOMEWORK you will be able to…



- Use the Ratio Test to determine whether a series converges or diverges
- Use the Root Test to determine whether a series converges or diverges.
- Know when to use what test!



THEOREM 9.17
RATIO TEST

Condition: Let $\sum a_n$ be a series with nonzero terms.

1. $\sum a_n$ converges absolutely if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|<1.$

2. $\sum a_n$ diverges if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|>1$ or $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\infty.$

3. The Ratio Test is inconclusive if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=1.$

USING THE RATIO TEST

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}$$

Does this series have nonzero terms?

yes.

$3^{n+1} = 3^n \cdot 3^1$

You got it! All terms are nonzero…that's a pretty easy condition to fulfill, don't you think?

Now we need to find a_n and a_{n+1}.

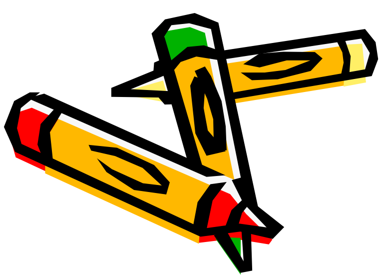
$$a_n = \frac{(-1)^n n}{3^n} \text{ and } a_{n+1} \frac{(-1)^{n+1}(n+1)}{3^{n+1}}$$

So let's make a ratio!

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(-1)^{n+1}(n+1)}{3^{n+1}}}{\dfrac{(-1)^n n}{3^n}}$$

$$= \frac{(-1)^{(n+1)-n}(n+1)}{3^{(n+1)-n}n}$$

$$= \frac{(-1)(n+1)}{3n}$$



Now for the test…

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)(n+1)}{3n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)}{3n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)/n}{3n/n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1+1/n}{3}\right|$$

$$= \frac{1}{3} < 1$$

Conclusion:

By the ratio test, this is an absolutely convergent series, and therefore convergent.

THEOREM 9.18
ROOT TEST

Condition: Let $\sum a_n$ be a series.

1. $\sum a_n$ converges absolutely if $\lim_{n\to\infty}\left(|a_n|\right)^{1/n} < 1$.

2. $\sum a_n$ diverges if $\lim_{n\to\infty}\left(|a_n|\right)^{1/n} > 1$ or $\lim_{n\to\infty}\left(|a_n|\right)^{1/n} = \infty$.

3. The Root Test is inconclusive if $\lim_{n\to\infty}\left(|a_n|\right)^{1/n} = 1$.

USING THE ROOT TEST

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{e^n}$$

Is this a series?

Yep!

Sure is! I like this condition best of all!

Now we need to find a_n and $\left(|a_n|\right)^{1/n}$.

$$\left(|a_n|\right)^{1/n} \to \sqrt[n]{|a_n|}$$

$$a_n = \frac{1}{e^n} \text{ and } \left(|a_n|\right)^{1/n} = \frac{1}{e}.$$

$$\left(\left|\frac{1}{e^n}\right|\right)^{1/n}$$

$$= \frac{1^{1/n}}{e^{n \cdot 1/n}}$$

$$= \frac{1}{e}$$

Now for the test…

$$\lim_{n\to\infty}\left(|a_n|\right)^{1/n} = \lim_{n\to\infty}\left(\left|\frac{1}{e^n}\right|\right)^{1/n}$$

$$= \lim_{n\to\infty}\left(\frac{1}{e}\right)$$

$$\approx \frac{1}{2.718\ldots} < 1$$

Conclusion:

By the root test, this is an absolutely convergent series, and therefore convergent.

WHICH TEST TO APPLY WHEN?!

Shhh…Let's talk about the series secrets

STEP 1

Apply the *n*th Term Test.

If $\lim_{n\to\infty} a_n \neq 0$, the infinite series

<u>diverges and you're all done!</u>

STEP 2

Check if the series is a geometric series or a *p*-series.

- If it's a geometric series with $0 < |r| < 1$, then it converges. Ex: $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n \rightarrow r = \frac{2}{3}$

- If it is a *p*-series with $p > 1$, then the series converges.

Ex: $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^2 = \sum_{n=1}^{\infty}\frac{1}{n^{2}} \rightarrow p = 2$

STEP 3

If $\sum_{n=1}^{\infty} a_n$ is a positive term series, use one of the following tests.

1. Direct Comparison Test
2. Limit Comparison Test

For these tests we <u>usually</u> compare with a geometric or p-series.

STEP 3 CONT.

3. Ratio Test: When there is an n!, or a c^n .
4. Root Test: When there is an n^n or some function of *n* to the *n*th power.
5. Integral Test: Must have terms a_n that are decreasing toward zero and that can be integrated.

STEP 4

If $\sum_{n=1}^{\infty} a_n$ is an alternating series, use one of the following tests.

1. Alternating Series Test.
2. A test from step 3 applied to $\sum_{n=1}^{\infty} |a_n|$. If $\sum_{n=1}^{\infty} |a_n|$ so does $\sum_{n=1}^{\infty} a_n$.