4/4/11

- Finish 10.4
- Lecture 10.5

Wednesday

Review

Friday

Exam 10.1-10.5

Hw due 10.1-10.5

MATH 251/GRACEY NAME________________________

WORKSHEET/10.4

POLAR COORDINATES AND POLAR GRAPHS

Describing a point in the plane in polar coordinates

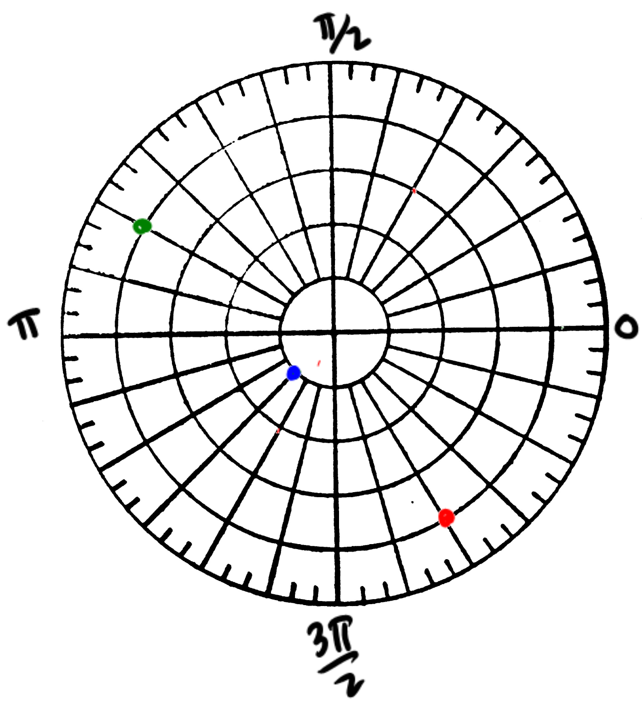
- Angles are in radians
- Points are described using the distance from the origin, called *r*, and its angle in radians measured **counterclockwise from the positive *x*-axis**, called θ.
    - This gives us the point (r, θ) instead of (x, y)
    - Since the angle measure is unique over one revolution of a circle, we do not need to use negative values for *r*
        - If we have a negative value for *r*, we simply add π to the angle measure and consider the positive value for *r*, so $(-r, \theta) = (r, \theta + \pi)$

1. Plot the point in polar coordinates and find the corresponding rectangular coordinates for the point.

a. $\left(1, \frac{5\pi}{4}\right)$

$\left(-4, \frac{11\pi}{6}\right)$

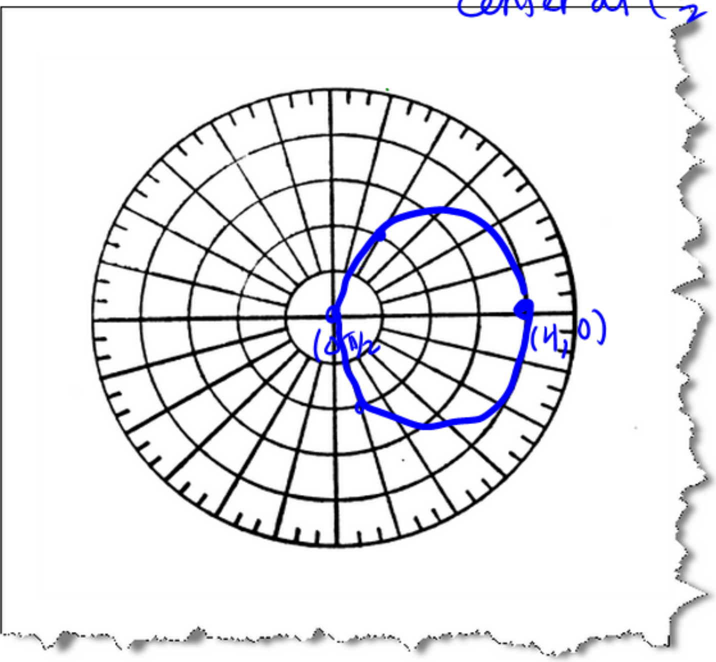
$= \left(4, \frac{5\pi}{6}\right)$



b. $\left(-3, \frac{2\pi}{3}\right)$

$f(\theta) = 4\cos\theta$

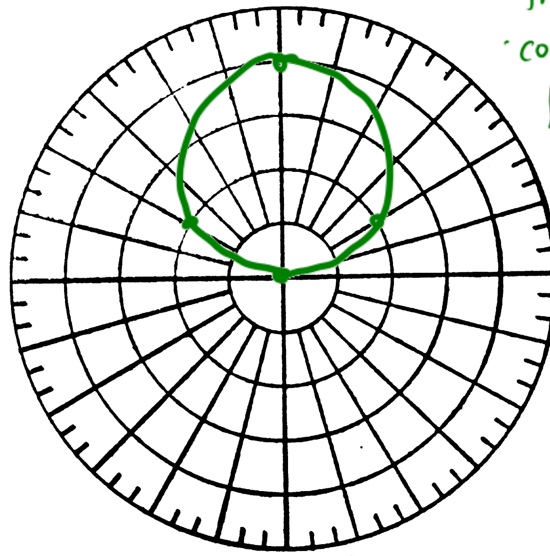
- circle that completes itself on ~~$[0,\pi]$~~ $[-\pi/2, \pi/2]$
- $r = a\cos\theta \rightarrow \frac{a}{2} = \text{radius}$
- sits on the y-axis
- center at $(\frac{a}{2}, 0)$



$r = a\sin\theta$

$g(\theta) = 4\sin\theta$

- circle that really sits on the x-axis
- completes in $[0, \pi]$
- $\frac{a}{2}$ = radius
- center $(\frac{a}{2}, \frac{\pi}{2})$

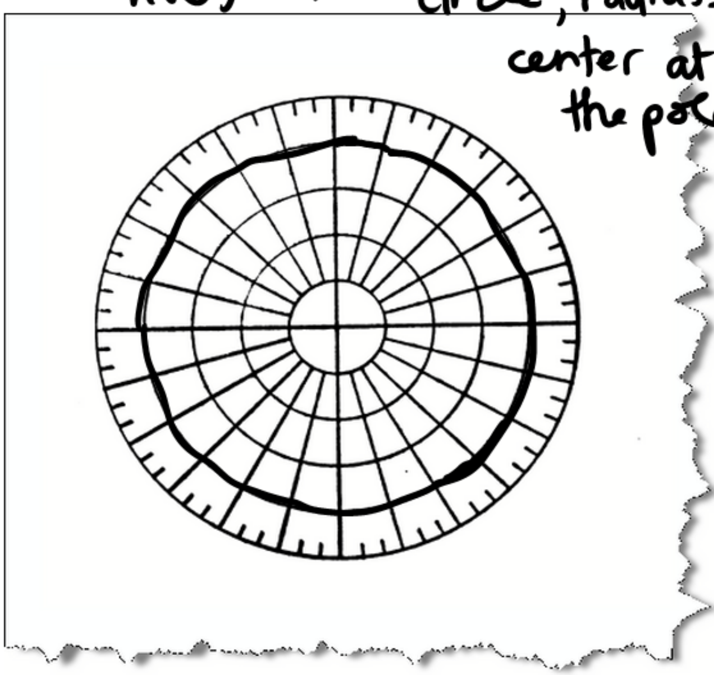


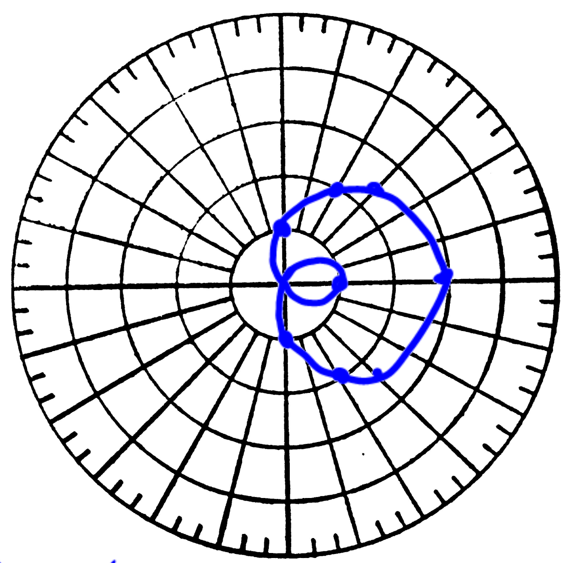
$r = a \pm b\cos\theta$

$f(\theta) = 1 + 2\cos\theta$

$h(\theta) = 4$

circle, radius = 4, center at the pole

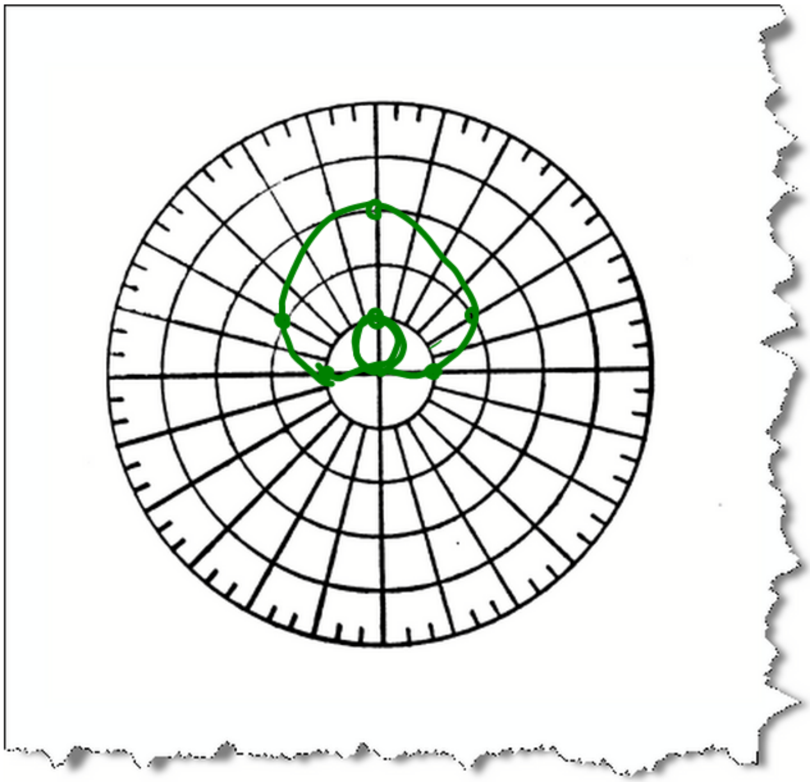




r	θ
3	0
1	$\pi/2$
-1	π
1	$3\pi/2$
2	$\pi/3$
2	$-\pi/3$
2.4	$\pi/4$

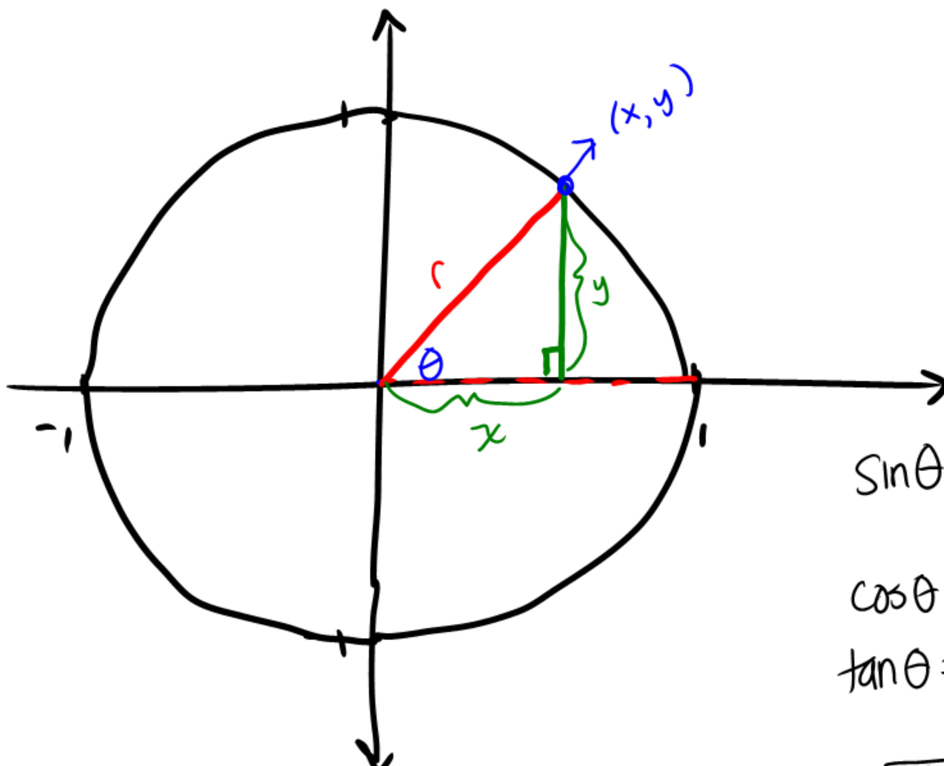
$\frac{a}{b} < 1$ (loop) $\quad \frac{a}{b} = 1$ cardioid (heart)

$1 < \frac{a}{b} < 2$ $\}\leftarrow$ dip $\quad \frac{a}{b} > 2$ $\{\leftarrow$ flat spot



$r = a \pm b\sin\theta$
$g(\theta) = 1 + 2\sin\theta$

r	θ
1	0
3	$\pi/2$
1	π
-1	$3\pi/2$



$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$

$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$

$\tan\theta = \frac{y}{x}$

$r = \sqrt{x^2 + y^2}$

Convert from rectangular to polar:

a) $(2, -4) \xrightarrow{\text{QIV}} (2\sqrt{5}, -1.1071)$

$r = \sqrt{(2)^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}$

$\tan\theta = -\frac{4}{2} = -2 \longrightarrow \arctan(-2) = \theta$

$\theta \approx -1.1071$

SLOPE IN POLAR FORM

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{f(\theta)\cos\theta+f'(\theta)\sin\theta}{-f(\theta)\sin\theta+f'(\theta)\cos\theta},\quad dx/d\theta\neq 0 \text{ at } (r,\theta)$$

$$\frac{y}{x}=\frac{r\sin\theta}{r\cos\theta}$$

$$\frac{\frac{\partial}{\partial\theta}f(\theta)\sin\theta}{\frac{\partial}{\partial\theta}f(\theta)\cos\theta}$$

$$=\frac{f(\theta)\cos\theta+f'(\theta)\sin\theta}{f(\theta)(-\sin\theta)+f'(\theta)\cos\theta}$$

Tangent lines at the pole

If $f(\alpha)=0$ and $f'(\alpha)\neq 0$, then the line $\theta=\alpha$ is tangent at the pole to the graph of $r=f(\theta)$.

Ex: Sketch a graph of the polar equation and find the tangents at the pole.

$r=3\cos 2\theta$

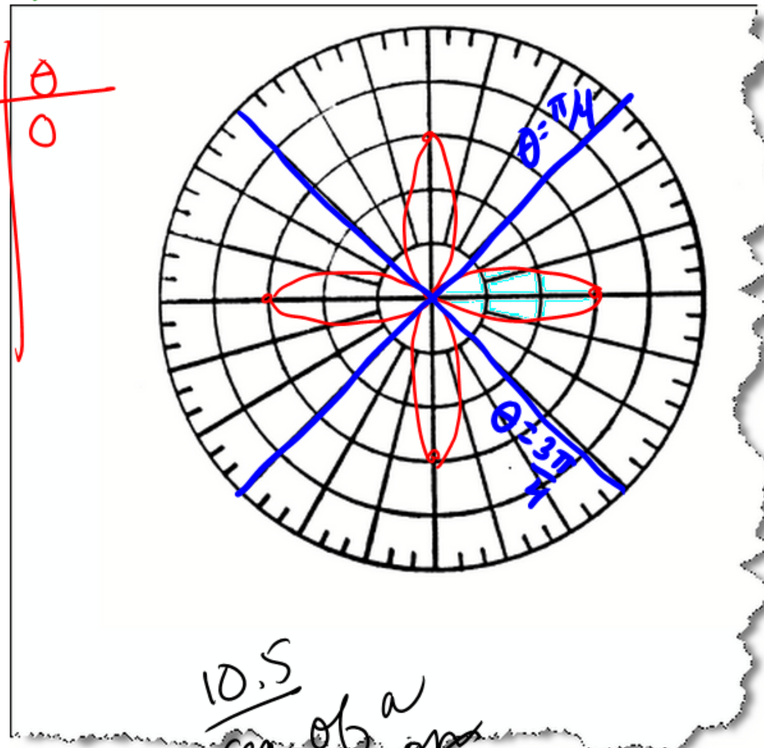
rose graph $\rightarrow$ $r=a\cos n\theta$ or $r=a\sin n\theta$

- If n is odd there's n petals
- If n is even there's 2n petals
- a is the length of each petal

$r = 3\cos 2\theta$

a) sketch

r	θ
3	0



b) Find tangents at the pole

1) Set $r=0$ and solve

$0 = 3\cos 2\theta$

$0 = \cos 2\theta$

$0 < \theta < 2\pi$
$0 < 2\theta < 4\pi$

$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ [these are the α's from the formula]

2) Find $\frac{dr}{d\theta}$ and check to make sure the $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$ don't zero out the derivative.

$\frac{dr}{d\theta} = -6\sin 2\theta$

at $\theta = \frac{\pi}{4}$: $\frac{dr}{d\theta} = -6\sin 2(\pi/4) = -6 \neq 0$

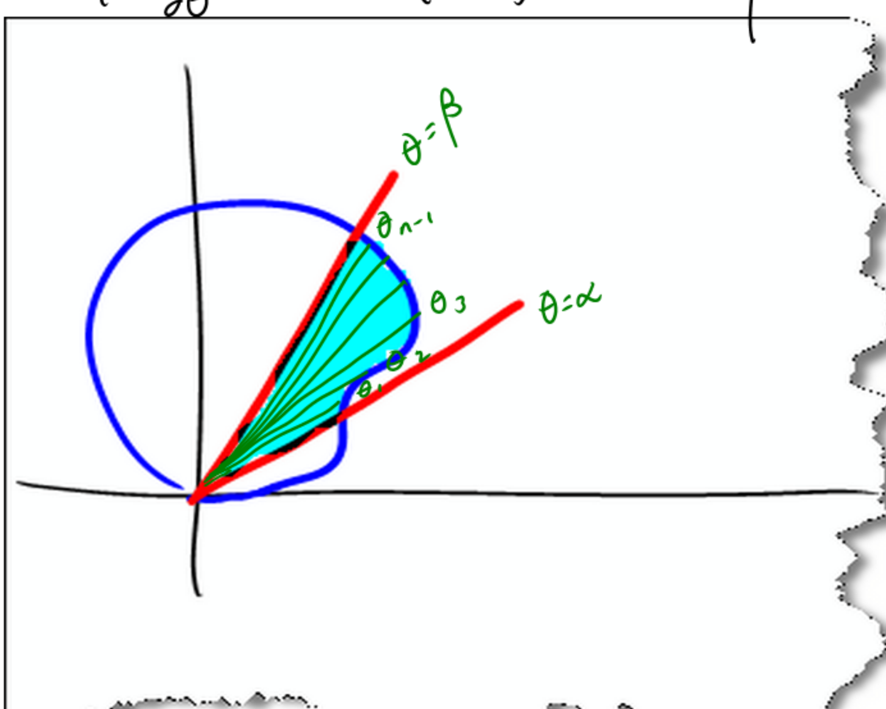
at $\theta = \frac{3\pi}{4}$ $\frac{dr}{d\theta} = -6\sin 2(3\pi/4) = 6 \neq 0$

10.5 Area of a polar region

$A_{\bigcirc} = \pi r^2$

$\theta = 2\pi$

$A_{sector} = \frac{1}{2}\theta r^2$



$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

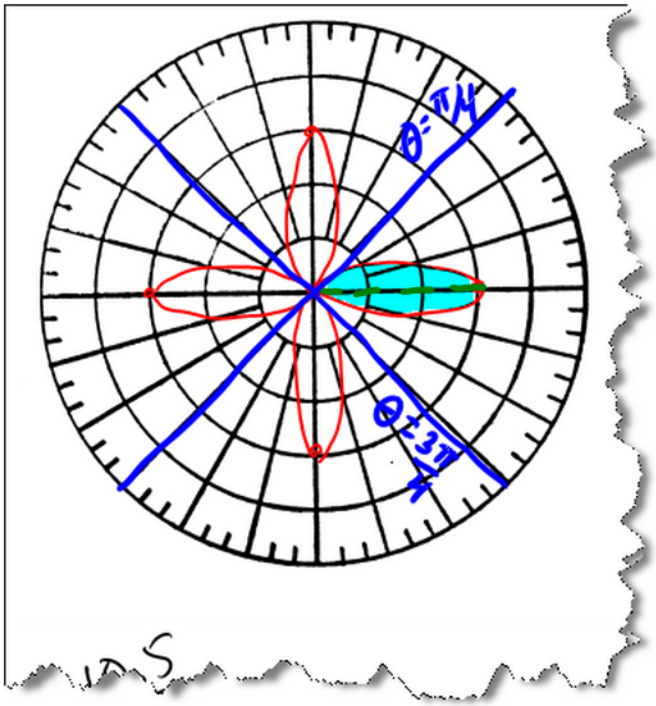
or

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$

$0 < \beta - \alpha \leq 2\pi$

Find the area of one petal of the rosegraph $r = 3\cos 2\theta$ (continuing the problem from before)

$r = 3\cos 2\theta$



$\cos^2 A = \frac{1+\cos 2A}{2}$

$\sin^2 A = \frac{1-\cos 2A}{2}$

petals are symmetric, so

$$A = 2\left[\frac{1}{2}\int_0^{\pi/4} [3\cos 2\theta]^2\, d\theta\right]$$

$$A = \int_0^{\pi/4} 9\cos^2 2\theta\, d\theta$$

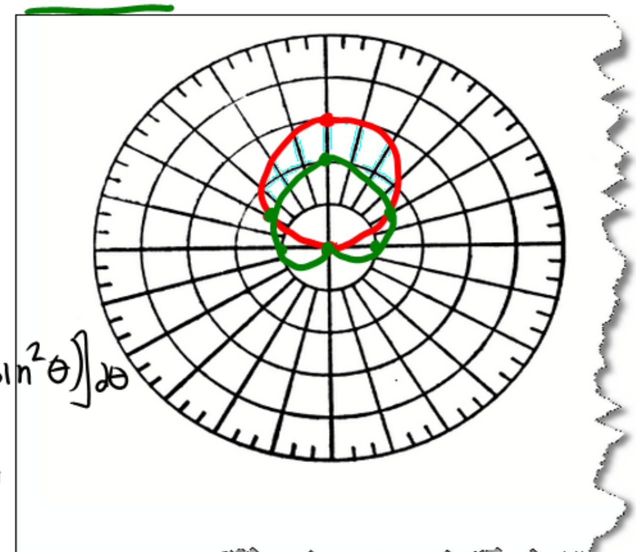
$$A = \frac{9}{2}\int_0^{\pi/4} (1+\cos 4\theta)\, d\theta$$

$$A = \frac{9}{2}\left(\theta + \frac{\sin 4\theta}{4}\right)\Big|_0^{\pi/4}$$

$$A = \frac{9}{2}\left[\left(\frac{\pi}{4}+0\right) - (0+0)\right]$$

$$\boxed{A = \frac{9\pi}{8} \text{ sq. units}}$$

Ex: Find the area of the region: Inside $r = 3\sin\theta$ and outside $r = 1+\sin\theta$



① Find the intersection angles

$3\sin\theta = 1+\sin\theta$

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$A = 2\cdot\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/2} (1+\sin\theta)^2 d\theta\right]$$

$$A = \int_{\pi/6}^{\pi/2} \left[9\sin^2\theta - (1+2\sin\theta+\sin^2\theta)\right] d\theta$$

$$A = \int_{\pi/6}^{\pi/2} (8\sin^2\theta - 2\sin\theta - 1)\, d\theta$$

$$A = \frac{8}{2}\int_{\pi/6}^{\pi/2}(1-\cos 2\theta) - \left[-2\cos\theta + \theta\right|_{\pi/6}^{\pi/2}$$

$$A = 4\left(\theta - \frac{\sin 2\theta}{2}\right|_{\pi/6}^{\pi/2} - \left[(0+\pi/2) - (-\sqrt{3} + \pi/6)\right]$$

$$A = 4\left[\left(\pi/2 - 0\right) - \left(\pi/6 - \frac{\sqrt{3}}{4}\right)\right] - \frac{\pi}{3} - \sqrt{3}$$

$$A = 4\left(\pi/3 + \frac{\sqrt{3}}{4}\right) - \frac{\pi}{3} - \sqrt{3}$$

$$A = \frac{4\pi}{3} + \sqrt{3} - \frac{\pi}{3} - \sqrt{3}$$

$$A = 3\pi/3$$

$$\boxed{A = \pi \text{ sq. units}}$$

ARC LENGTH IN POLAR FORM

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The **length of the graph** of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

AREA OF A SURFACE OF REVOLUTION

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The **area of the surface** formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line is as follows.

1. $S = 2\pi \int_\alpha^\beta f(\theta)\sin\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$ (about the polar axis)

2. $S = 2\pi \int_\alpha^\beta f(\theta)\cos\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$ $\left(\text{about the line } \theta = \frac{\pi}{2}\right)$