1/28/11

- warm up using 5.8 worksheet
- lecture 5.8

Monday

- Finish 5.8

Wednesday

Review

Next Friday

- Exam1/Ch5.6-5.8
- Review HW and 5.6-5.8 HW is due

When you are done with your homework you should be able to…

- π Develop properties of hyperbolic functions
- π Differentiate and integrate hyberbolic functions
- π Develop properties of inverse hyperbolic functions
- π Differentiate and integrate functions involving inverse hyperbolic functions

Warm-up: Find the following definite and indefinite integrals.

a. $\int \frac{x-1}{\sqrt{x^2-2x}}dx = \int (\cancel{x-1})\, u^{-1/2}\left(\frac{du}{2(\cancel{x-1})}\right)$

$u = x^2-2x$

$\frac{du}{dx} = 2x-2$

$dx = \frac{du}{2(x-1)}$

$= \frac{1}{2}\int u^{-1/2}\,du$

$= \frac{\cancel{1}}{\cancel{2}}\frac{u^{1/2}}{\cancel{1/2}} + C$

$= \boxed{(x^2-2x)^{1/2} + C}$

b. $\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}}dx = \int_{\pi/2}^{\pi/4} \frac{u}{\cancel{\sqrt{1-x^2}}}\left(-\cancel{\sqrt{1-x^2}}\,du\right) = -\frac{u^2}{2}\Big|_{\pi/2}^{\pi/4}$

$u = \arccos x$

$\frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$

$dx = -\sqrt{1-x^2}\,du$

upper lim: $\arccos\frac{1}{\sqrt{2}} = \pi/4$

lower lim: $\arccos 0 = \pi/2$

$= -\frac{1}{2}\left(\frac{\pi^2}{16} - \frac{4}{4}\cdot\frac{\pi^2}{4}\right)$

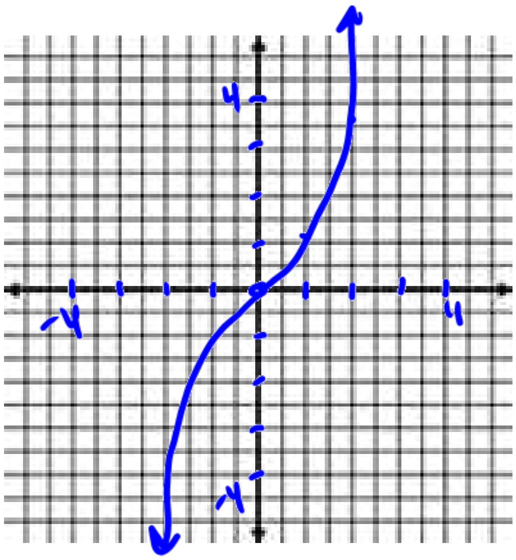
$= -\frac{1}{2}\left(\frac{\pi^2 - 4\pi^2}{16}\right)$

$= -\frac{1}{2}\left(-\frac{3\pi^2}{16}\right)$

$= \boxed{\frac{3\pi^2}{32}}$

1. Graph $f(x)=\dfrac{e^{x}-e^{-x}}{2}$.

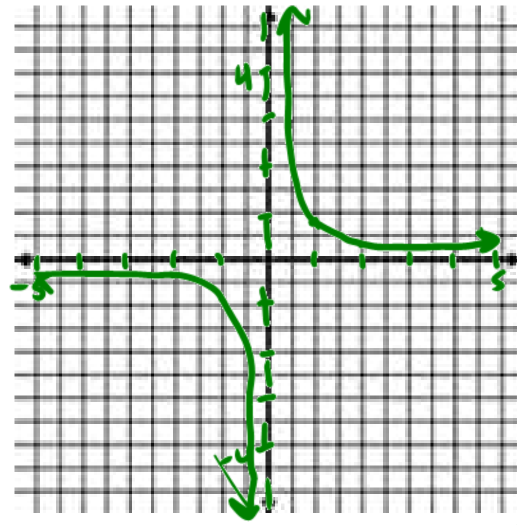
$f(x)=\sinh x$



x	$f(x)$
0	0
1	1.2
2	3.6
3	10
-1	-1.2
-2	-3.6

2. Graph $g(x)=\dfrac{2}{e^{x}-e^{-x}}$. $, x\neq 0$

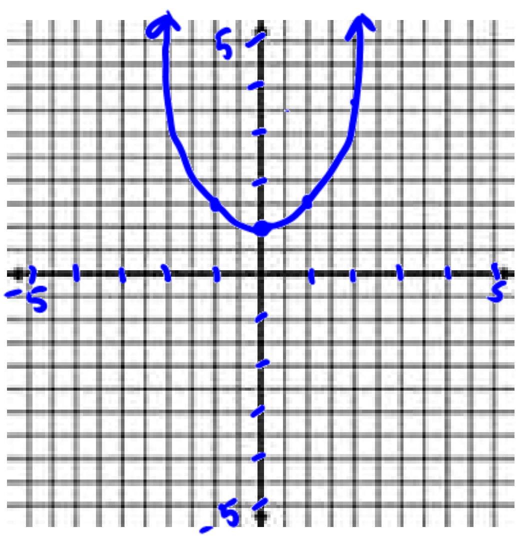
$g(x)=\operatorname{csch} x$



x	$g(x)$
-2	-.3
-1	-.9
-0.01	-99
1	.9
2	.3
0.01	99
-10	0
10	0

3. Graph $f(x)=\dfrac{e^{x}+e^{-x}}{2}$.

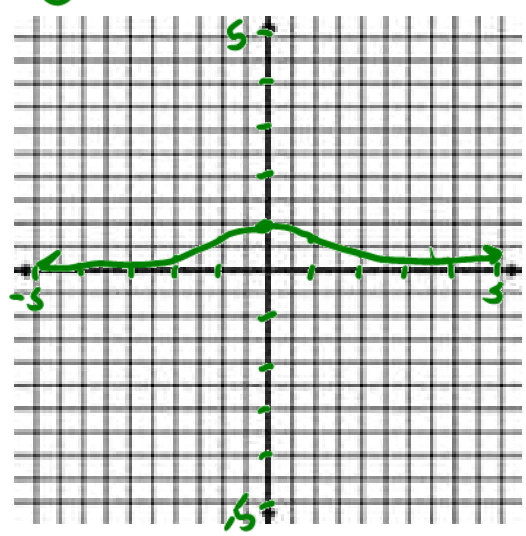
$f(x)=\cosh x$



x	$f(x)$
0	1
1	1.5
2	3.7
-1	1.5
-2	3.7

4. Graph $g(x)=\dfrac{2}{e^{x}+e^{-x}}$.

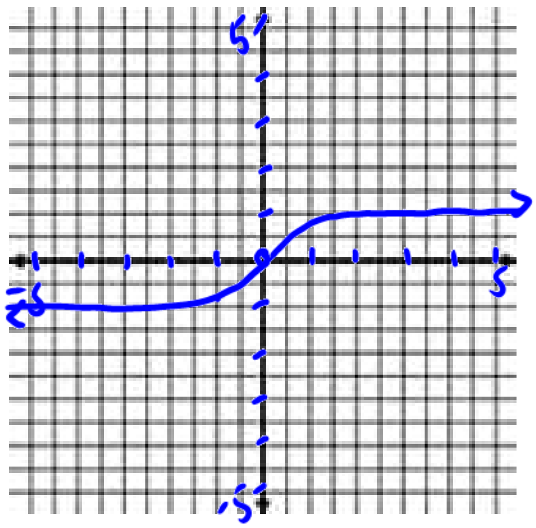
$g(x)=\operatorname{sech} x$



x	$g(x)$
0	1
1	.7
2	.3
10	0
-1	.7
-2	.3

5. Graph $f(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$.

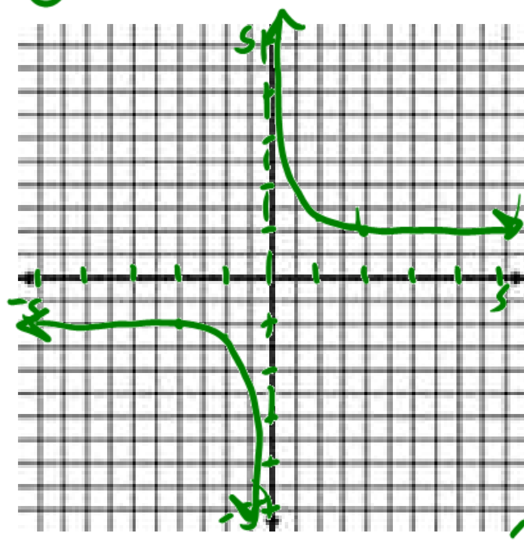
$f(x) = \tanh x$



x	$f(x)$
0	0
1	.8
2	1
-1	-.8
-2	-1
10	1
-10	-1

6. Graph $g(x)=\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$.

$g(x) = \coth x$



x	$g(x)$
1	1.3
2	1
-1	-1.3
-2	-1
10	1
-10	-1
-0.01	-100
0.01	100

DEFINITIONS OF THE HYPERBOLIC FUNCTIONS

$$\sinh x = \frac{e^x-e^{-x}}{2} \qquad \operatorname{csch} x = \frac{2}{e^x-e^{-x}} = \frac{1}{\sinh x},\ x \neq 0$$

$$\cosh x = \frac{e^x+e^{-x}}{2} \qquad \operatorname{sech} x = \frac{2}{e^x+e^{-x}} = \frac{1}{\cosh x},\ x \neq 0$$

$$\tanh x = \frac{e^x-e^{-x}}{e^x+e^{-x}} \qquad \coth x = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{1}{\tanh x},\ x \neq 0$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{(e^x-e^{-x})/\cancel{2}}{(e^x+e^{-x})/\cancel{2}}$$

Example 1: Evaluate each function.

a. $\cosh 0 = \dfrac{e^{0}+e^{-0}}{2} = \dfrac{1+1}{2} = \boxed{1}$

$\boxed{e^{-\ln 3} = e^{\ln 3^{-1}} = \frac{1}{3}}$

b. $\tanh(\ln 3) = \dfrac{e^{\ln 3} - e^{-(\ln 3)}}{e^{\ln 3} + e^{-(\ln 3)}} = \dfrac{3-\frac{1}{3}}{3+\frac{1}{3}} = \dfrac{\frac{8}{\cancel{3}}}{\frac{10}{\cancel{3}}} = \boxed{\frac{4}{5}}$

Example 2: Verify the identity.

$(a+b)^2 = a^2+2ab+b^2$

$$\coth^2 x - \operatorname{csch}^2 x = 1$$

$$\left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)^2 - \left(\frac{2}{e^x-e^{-x}}\right)^2 = 1$$

$$\frac{(e^x+e^{-x})^2-(2)^2}{(e^x-e^{-x})^2} = 1$$

$$\frac{(e^x)^2+2(e^x)(e^{-x})+(e^{-x})^2-4}{(e^x-e^{-x})^2} = 1$$

$$e^{2x}+2e^0+e^{-2x}-4 = (e^x-e^{-x})^2$$

$$e^{2x}-2+e^{-2x} = e^{2x}-2+e^{-2x} \;//$$

HYPERBOLIC IDENTITIES

$\cosh^2 x - \sinh^2 x = 1$	$\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$
$\tanh^2 x + \operatorname{sech}^2 x = 1$	$\sinh(x-y) = \sinh x\cosh y - \cosh x\sinh y$
$\coth^2 x - \operatorname{csch}^2 x = 1$	$\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$
$\sinh^2 x = \dfrac{-1+\cosh 2x}{2}$	$\cosh(x-y) = \cosh x\cosh y - \sinh x\sinh y$
$\cosh^2 x = \dfrac{1+\cosh 2x}{2}$	
$\sinh 2x = 2\sinh x\cosh x$	
$\cosh 2x = \cosh^2 x + \sinh^2 x$	

Example 3: Differentiate with respect to *x*.

a. $\frac{\partial}{\partial x} y = \frac{\partial}{\partial x}\left(\frac{e^x - e^{-x}}{2}\right)$ sinhx

$$\frac{dy}{dx} = \frac{1}{2}\left(e^x - \left(-e^{-x}\right)\right)$$

$$\frac{dy}{dx} = \frac{e^x + e^{-x}}{2}$$

$$\boxed{\frac{dy}{dx} = \cosh x}$$

b. $\frac{\partial}{\partial x} y = \frac{\partial}{\partial x}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$ ← tanh x

$$\frac{dy}{dx} = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$\frac{dy}{dx} = \frac{\cancel{e^{2x}} + 2e^x e^{-x} + \cancel{e^{-2x}} - (\cancel{e^{2x}} - 2e^x e^{-x} + \cancel{e^{-2x}})}{(e^x + e^{-x})^2}$$

$$\frac{dy}{dx} = \frac{4}{(e^x + e^{-x})^2} \longrightarrow \operatorname{sech}^2 x$$

Example 4: Find the integral.

a. $\int \frac{e^x + e^{-x}}{2} dx$

b. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$

THEOREM: DERIVATIVES AND INTEGRALS OF HYPERBOLIC FUNCTIONS

$\frac{d}{dx}[\sinh u] = (\cosh u)u'$	$\int \cosh u\,du = \sinh u + C$
$\frac{d}{dx}[\cosh u] = (\sinh u)u'$	$\int \sinh u\,du = \cosh u + C$
$\frac{d}{dx}[\tanh u] = (\operatorname{sech}^2 u)u'$	$\int \operatorname{sech}^2 u\,du = \tanh u + C$
$\frac{d}{dx}[\coth u] = -(\operatorname{csch}^2 u)u'$	$\int \operatorname{csch}^2 u\,du = -\coth u + C$
$\frac{d}{dx}[\operatorname{sech} u] = -(\operatorname{sech} u \tanh u)u'$	$\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$
$\frac{d}{dx}[\operatorname{csch} u] = -(\operatorname{csch} u \coth u)u'$	$\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$

Example 5: Differentiate with respect to *x*.

a. $f(x) = \tanh(3x^2 - 2)$

b. $y = \ln(\cosh x)$

Example 6: Find the integral.

a. $\int \frac{\cosh \sqrt{x}}{\sqrt{x}} dx$

b. $\int \frac{\sinh x}{1+\sinh^2 x} dx$

INVERSE HYPERBOLIC FUNCTIONS

FUNCTION	DOMAIN
1. $\sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$	$(-\infty,\infty)$
2. $\cosh^{-1} x = \ln\left(x+\sqrt{x^2-1}\right)$	$[1,\infty)$
3. $\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$	$(-1,1)$
4. $\coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1}$	$(-\infty,-1)\cup(1,\infty)$
5. $\operatorname{sech}^{-1} x = \ln\frac{1+\sqrt{1-x^2}}{x}$	$(0,1]$
6. $\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x}+\frac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$	$(-\infty,0)\cup(0,\infty)$

Example 7: Evaluate each function.

a. $\sinh^{-1} 0$

b. $\operatorname{csch}^{-1} \frac{2}{3}$

DERIVATIVES AND INTEGRALS OF INVERSE HYPERBOLIC FUNCTIONS

$$\frac{d}{dx}[\sinh u]=(\cosh u)u'$$

$$\frac{d}{dx}[\cosh u]=(\sinh u)u'$$

$$\frac{d}{dx}[\tanh u]=\left(\operatorname{sech}^2 u\right)u'$$

$$\frac{d}{dx}[\coth u]=-\left(\operatorname{csch}^2 u\right)u'$$

$$\frac{d}{dx}[\operatorname{sech} u]=-(\operatorname{sech} u\tanh u)u'$$

$$\frac{d}{dx}[\operatorname{csch} u]=-(\operatorname{csch} u\coth u)u'$$

$$\frac{d}{dx}\left[\sinh^{-1} u\right]=\frac{u'}{\sqrt{u^2+1}}$$

$$\frac{d}{dx}\left[\cosh^{-1} u\right]=\frac{u'}{\sqrt{u^2-1}}$$

$$\frac{d}{dx}\left[\tanh^{-1} u\right]=\frac{u'}{1-u^2}$$

$$\frac{d}{dx}\left[\operatorname{sech}^{-1} u\right]=\frac{-u'}{u\sqrt{1-u^2}}$$

$$\frac{d}{dx}\left[\coth^{-1} u\right]=\frac{u'}{1-u^2}$$

$$\frac{d}{dx}\left[\operatorname{csch}^{-1} u\right]=\frac{-u'}{|u|\sqrt{1+u^2}}$$

$$\int\frac{du}{\sqrt{u^2\pm a^2}}=\ln\left(u+\sqrt{u^2\pm a^2}\right)+C$$

$$\int\frac{du}{a^2-u^2}=\frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right|+C$$

$$\int\frac{du}{u\sqrt{a^2\pm u^2}}=-\frac{1}{a}\ln\frac{a+\sqrt{a^2\pm u^2}}{|u|}+C$$

Example 8: Find the derivative of the function. Simplify your result to a single rational expression with positive exponents.

a. $f(x) = \coth^{-1} x^2$

b. $g(x) = x \tanh^{-1} x + \ln\sqrt{1-x^2}$

c. $y = \operatorname{sech}^{-1}(\cos 2x), \quad 0 < x < \dfrac{\pi}{4}$

Example 9: Find the limit.

a. $\lim_{x \to -\infty} \sinh x$

b. $\lim_{x \to 0^-} \coth x$

Example 10: Find the integral.

a. $\int \frac{1}{2x\sqrt{1-4x^2}}dx$

b. $\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}}$

c. $\int \frac{3}{\sqrt{x}\sqrt{9+x}}dx$

d. $\int \frac{1}{1-4x-2x^2}dx$