THEOREM: THE CONSTANT RULE

The derivative of a constant function is zero. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0$$

Example 1: Find the derivative of the function $g(x) = -5$.

$g'(x) = 0$

THEOREM: THE POWER RULE

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing zero.

Example 2: Find the following derivatives.

a. $f(x) = x^{-5}$

$f'(x) = -5x^{-6}$

$f'(x) = -\frac{5}{x^6}$

b. $f(x) = x^{1/2}$

$f'(x) = 1/2\, x^{-1/2}$

$= \frac{1}{2\sqrt{x}}$

c. $f(x) = x^{-2/3}$

$f'(x) = -\frac{2}{3} x^{-5/3}$

$= -\frac{2}{3\sqrt[3]{x^5}}$

THEOREM: THE CONSTANT MULTIPLE RULE

If f is a differentiable function and c is a real number, then cf is also differentiable and

$$\frac{d}{dx}\left[cf(x)\right] = cf'(x)$$

Example 3: Find the slope of the graph of $f(x)=2x^3$ at

$f'(x)=2\cdot 3x^2$

$f'(x)=6x^2$

a. $x=2$

$f'(2)=6(2)^2$

$f'(2)=24$

b. $x=-6$

$f'(-6)=216$

c. $x=0$

$f'(0)=0$

THEOREM: THE SUM AND DIFFERENCE RULES

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f+g$ (or $f-g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}\left[f(x)+g(x)\right]=f'(x)+g'(x)$$

$$\frac{d}{dx}\left[f(x)-g(x)\right]=f'(x)-g'(x)$$

Example 4: Find the equation of the line tangent to the graph of $f(x)=x-\sqrt{x}$ at $x=4$.

$f'(x)=1-\frac{1}{2}x^{-1/2}$

$f'(4)=1-\frac{1}{2\sqrt{4}}$

$f'(4)=1-\frac{1}{4}$

$f'(4)=\frac{3}{4}$

THEOREM: DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\sin x]=\cos x \qquad \frac{d}{dx}[\cos x]=-\sin x$$

$$\frac{d}{dx}[\csc x]=-\csc x\cot x \qquad \frac{d}{dx}[\sec x]=\sec x\tan x$$

$$\frac{d}{dx}[\tan x]=\sec^2 x \qquad \frac{d}{dx}[\cot x]=-\csc^2 x$$

Example 5: Find the derivative of the following functions:

a. $f(x)=\dfrac{\sin x}{6}$

$f'(x)=\frac{1}{6}(\cos x)$

$=\frac{\cos x}{6}$

b. $r(\theta)=5x-3\cos\theta$

$r'(\theta)=5+3\sin\theta$

c. $f(x)=x\tan x$

$f'(x)=1\tan x+x\sec^2 x$

$f'(x)=\tan x+x\sec^2 x$

d. $r(\theta)=\dfrac{\cos\theta}{\cot\theta}$

TRIG Identities!

$r(\theta)=\frac{\cos\theta}{\left(\frac{\cos\theta}{\sin\theta}\right)}$ OR $=\sin\theta$

$r'(\theta)=\frac{\cot\theta(-\sin\theta)-\left[\cos\theta(-\csc^2\theta)\right]}{\cot^2\theta}$

$r'(\theta)=\frac{-\cot\theta\sin\theta+\cos\theta\csc^2\theta}{\cot^2\theta}$

$r'(\theta)=\cos\theta$

THEOREM: THE PRODUCT RULE

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the derivative of the first function times the second function, plus the first function times the derivative of the second function.

$$\frac{d}{dx}\left[f(x)g(x)\right]=f'(x)g(x)+f(x)g'(x)$$

This rule extends to cover products of more than two factors. For example the derivative of the product of functions $fghk$ is

$$\frac{d}{dx}[fghk]=f'(x)g(x)h(x)k(x)+f(x)g'(x)h(x)k(x)+f(x)g(x)h'(x)k(x)+f(x)g(x)h(x)k'(x)$$

Example 6: Find the derivative of the following functions. Simplify your result to a single rational expression with positive exponents.

a. $g(x)=x\cos x$

$g'(x)=(1)(\cos x)+(x)(-\sin x)$

$=\cos x-x\sin x$

b. $h(t)=\left(3-\sqrt{t}\right)^2$

u^2 $\quad u = 3-\sqrt{t}$

$2u \; du/dx \quad du/dx = -\frac{1}{2\sqrt{t}}$

$2u \cdot -\frac{1}{2\sqrt{t}} \quad -\frac{u}{\sqrt{t}} = -\frac{(3-\sqrt{t})}{\sqrt{t}}$

c. $f(x)=\left(x^3-x\right)\left(x^2+2\right)\left(x^2+x-1\right)$

$f'(x)=(3x^2-1)(x^2+2)(x^2+x-1)+(x^3-x)(2x)(x^2+x-1)+(x^3-x)(x^2+2)(2x+1)$

THEOREM: THE QUOTIENT RULE

The quotient of two differentiable functions f and g is itself differentiable at all values of x for which $g(x)\neq 0$. Moreover, the derivative of f/g is the derivative of the numerator times the denominator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2}$$

Example 7: Find the derivative of the following functions. Simplify your result to a single rational expression with positive exponents.

a. $g(x)=x^4\left(1-\frac{2}{x+1}\right) = x^4\left(1-2(x+1)^{-1}\right)$

$g'(x)=4x^3\left(1-2(x+1)^{-1}\right)+x^4\left(0+2(x+1)^{-2}(1)\right)$

$g'(x)=4x^3-8x^3(x+1)^{-1}+2x^4(x+1)^{-2}$

$g'(x)=2x^3(x+1)^{-2}\left[2(x+1)^2-4(x+1)+x\right]$

$$g'(x) = \frac{2x^3}{(x+1)^2}\left[2(x^2+2x+1) - 4x - 4 + x\right]$$

$$g'(x) = \frac{2x^3}{(x+1)^2}\left[2x^2+4x+2-3x-4\right]$$

$$\boxed{g'(x) = \frac{2x^3(2x^2+x-2)}{(x+1)^2}}$$

b. $h(s)=\frac{s}{\sqrt{s}-1}$

$h'(s)=\frac{1(\sqrt{s}-1)\frac{2\sqrt{s}}{2\sqrt{s}}-s\left(\frac{1}{2\sqrt{s}}\right)}{(\sqrt{s}-1)^2}$

$h'(s)=\frac{2s-2\sqrt{s}-s}{2\sqrt{s}(\sqrt{s}-1)^2}$

$h'(s)=\frac{s-2\sqrt{s}}{2\sqrt{s}(\sqrt{s}-1)^2}$

$h'(s)=\frac{\sqrt{s}-2}{2(\sqrt{s}-1)^2}$

c. $f(x)=\tan x$

$f'(x)=\sec^2 x$

Example 8: Find the derivative of the trigonometric functions.

a. $g(x)=-2\csc x$

b. $h(t)=\cot^2 t$

c. $r(s)=\frac{\sec s}{s}$

Example 9: Find the given higher-order derivative.

a. $f''(x) = 2 - \frac{2}{x}, \quad f'''(x)$

b. $f^{(4)}(x) = 2x + 1, \quad f^{(6)}(x)$

Theorem: The Chain Rule

If $y = f(u)$ is a differentiable function of *u* and $u = g(x)$ is a differentiable function of *x*, then $y = f(g(x))$ is a differentiable function of *x* and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \text{ or } \frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\,g'(x).$$

Example 10: Find the derivative using the Chain Rule.

a. $y = (2 - x)^3$

b. $f(x) = \sin 2x$

c. $h(t) = \dfrac{\sqrt{t}}{\sqrt{t} - 1}$

Theorem: The General Power Rule

If $y = \left[u(x)\right]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n\left[u(x)\right]^{n-1} \cdot \frac{du}{dx} \text{ or } \frac{d}{dx}\left[u^n\right] = nu^{n-1}u'.$$

Example 11: Find the derivative of the following functions.

a. $y = \sec x$

b. $y = \sec 2x$

c. $y = \sec^2 x$

d. $y = \sec x^2$

e. $y = x^5$

f. $y = \left(2x^3 - 5\right)^5$

g. $y = \sqrt{x}$

h. $y = \sqrt{\cos x}$

i. $f(x)=x^2(2-x)^{2/3}$

j. $f(x)=\sqrt{\dfrac{1}{2x^3+15}}$

k. $h(x)=x\sin^2 4x$

l. $f(x)=\cot\sqrt[3]{x}-\sqrt[3]{\cot x}$

Example 12: Find the equation of the tangent line at $t=1$ for the function

$$s(t)=\left(9-t^2\right)^{2/3}.$$

Theorem: Derivative of the Natural Logarithmic Function

Let *u* be a differentiable function of *x*.

1. $\frac{d}{dx}[\ln x]=\frac{1}{x}$

2. $\frac{d}{dx}[\ln u]=\frac{u'}{u},\quad u>0$

Example 13: Find the derivative.

a. $y=(\ln x)^3$

b. $f(x)=\ln|\cos x|$

c. $h(t)=\ln x^x$

Theorem: Derivative of the Natural Exponential Function

Let u be a differentiable function of x.

1. $\frac{d}{dx}\left[e^x\right]=e^x$
2. $\frac{d}{dx}\left[e^u\right]=e^u u'$

Example 14: Find the derivative.

a. $y=xe^{-x}$

b. $f\left(x\right)=e^{\sin 2x}$

c. $h\left(t\right)=\frac{e^t}{\ln e^{\sqrt{t}}}$

Theorem: Derivatives for Bases other than *e*

Let a be a positive real number $(a \neq 1)$ and let u be a differentiable function of x.

1. $\frac{d}{dx}\left[a^x\right] = (\ln a)\,\mathrm{a}^x$
2. $\frac{d}{dx}\left[a^u\right] = (\ln a)\,\mathrm{a}^u u'$
3. $\frac{d}{dx}\left[\log_a x\right] = \frac{1}{(\ln a)\,x}$
4. $\frac{d}{dx}\left[\log_a u\right] = \frac{u'}{(\ln a)\,u}$

Example 15: Find the derivative.

a. $y = 2^{3x}$

b. $f(x) = \log 5x$

c. $h(t) = \frac{\log_3 t^2}{\sin t}$