MATH 270/GRACEY NAME Key

EXAM 3/SECTIONS 5.1-5.4, 6.1-6.2 HW______% Q______% O______% E______%

100 POINTS POSSIBLE

YOUR WORK MUST BE ORGANIZED AND CLEAR

1. (10 POINTS, 2 POINTS EACH) Please circle true or false.

a. T F The Cauchy-Schwarz Inequality states that if If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then $|\mathbf{u}\cdot\mathbf{v}| \geq \|\mathbf{u}\|\|\mathbf{v}\|$.

b. T F $c\langle \mathbf{u},\mathbf{v}\rangle = \langle c\mathbf{u}, c\mathbf{v}\rangle$.

c. T F A set S of vectors is orthogonal when every pair of vectors in S is orthogonal.

d. T F An orthonormal basis derived by the Gram-Schmidt orthonormalization process does not depend on the order of the vectors in the basis.

e. T F A linear transformation *T* from V into W is one-to-one when the preimage of every $\mathbf{w}$ in the range consists of a single vector $\mathbf{v}$.

2. (10 POINTS, 2 POINTS PER BLANK) Please fill in the blank.

a. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space V, then S is linearly independent.

b. Let $T: V \to W$ be a linear transformation. The dimension of the kernel of *T* is called the nullity of *T*.

c. Let $T: R^n \to R^m$ be a linear transformation given by $T(A) = A\mathbf{x}$. Then the column space of A is equal to the range of *T*.

3. (5 POINTS) Determine all vectors that are orthogonal to $(1,-1,2)$.

$\vec{u} = (u_1, u_2, u_3)$

$(u_1, u_2, u_3) \cdot (1, -1, 2) = 0$

$u_1 - u_2 + 2u_3 = 0 \rightarrow u_1 = u_2 - 2u_3$

Let $u_2 = s$, Let $u_3 = t$

$u_1 = s - 2t$

$\vec{u} = (s-2t, s, t)$ where s and $t \in \mathbb{R}$

4. (6 POINTS) Let $f(x) = x$, $g(x) = \dfrac{1}{x^2+1}$, and $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$.

a. (5 POINTS) Find the inner product.

$\langle f, g \rangle = \frac{1}{2} \int_{-1}^{1} 2x \left(\frac{1}{x^2+1}\right) dx$

$= \frac{1}{2} \ln(x^2+1) \Big|_{-1}^{1}$

$= \frac{1}{2} \left[\ln(2) - \ln(2) \right]$

$= \boxed{0}$

b. (1 POINTS) Are the vectors are orthogonal?

Circle One: YES NO

5. (10 POINTS) Let $B = \{(-1,2,2),(1,0,0)\}$ be a basis for a subspace of R^3, and let $\mathbf{x} = \{(-3,4,4)\}$ be a vector in the subspace.

a. (3 POINTS) Find the coordinates of $\mathbf{x}$ relative to B.

We need to find $\vec{x} = (-3,4,4)$ as a linear combo of the vectors in B.

$c_1(-1,2,2) + c_2(1,0,0) = (-3,4,4)$

$-c_1 + c_2 = -3$
$2c_1 = 4 \rightarrow c_1 = 2, \; c_2 = -1$ so $\boxed{[\vec{x}]_B = (2,-1)}$
$2c_1 = 4$

b. (6 POINTS) Apply the Gram-Schmidt orthonormalization process to transform B into an orthonormal set B'.

$\vec{w}_1 = (-1,2,2)$

$\vec{w}_2 = (1,0,0) - \dfrac{\langle (1,0,0),(-1,2,2)\rangle}{\langle (-1,2,2),(-1,2,2)\rangle} \cdot (-1,2,2)$

$\vec{w}_2 = (1,0,0) - \frac{-1}{9}(-1,2,2)$

$\vec{w}_2 = \frac{1}{9}(8,2,2)$

$\vec{w}_2 = \frac{2}{9}(4,1,1)$

$\|\frac{2}{9}(4,1,1)\| = |\frac{2}{9}|\sqrt{18}$
$= \frac{2}{9} \cdot 3\sqrt{2}$

$B' = \left\{ \dfrac{(-1,2,2)}{\|(-1,2,2)\|}, \dfrac{\frac{2}{9}(4,1,1)}{\|\frac{2}{9}(4,1,1)\|} \right\} = \boxed{\left\{ \underbrace{\left(-\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right)}_{\vec{u}_1}, \underbrace{\left(\tfrac{4}{3\sqrt{2}}, \tfrac{1}{3\sqrt{2}}, \tfrac{1}{3\sqrt{2}}\right)}_{\vec{u}_2} \right\}}$

c. (3 POINTS) Find the coordinates of $\mathbf{x}$ relative to B'.

$\boxed{[\vec{x}]_{B'} = \left(\tfrac{19}{3}, -\tfrac{4}{3\sqrt{2}}\right)}$

$\langle x, \vec{u}_1\rangle = (-3,4,4)\cdot\left(-\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right)$
$= \frac{19}{3}$

$\langle x, \vec{u}_2\rangle = (-3,4,4)\cdot\left(\tfrac{4}{3\sqrt{2}}, \tfrac{1}{3\sqrt{2}}, \tfrac{1}{3\sqrt{2}}\right)$
$= -\frac{4}{3\sqrt{2}}$

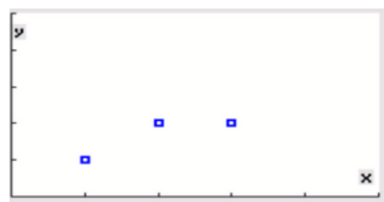
6. (5 POINTS) Prove that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^2, then $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$.

Proof: Let $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$, and $\vec{w} = (w_1, w_2)$

$$
\begin{aligned}
\text{So } (\vec{u}+\vec{v})\cdot\vec{w} &= (u_1+v_1, u_2+v_2)\cdot(w_1, w_2) \\
&= (u_1+v_1)w_1 + (u_2+v_2)w_2 && \text{(Defn of dot product)} \\
&= u_1w_1 + v_1w_1 + u_2w_2 + v_2w_2 && \text{(Real \#'s are distributive)} \\
&= u_1w_1 + u_2w_2 + v_1w_1 + v_2w_2 && \text{(Real \#'s are commutative)} \\
&= \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}. \checkmark
\end{aligned}
$$

7. (10 POINTS) We are collecting data on the number of machine failures per day in some factory. We collected three data points in the form (day, number of failures). The three data points are (1, 1), (2, 2), and (3, 2).

a. Plot the data. Is there a linear trend?
Circle One: YES NO



b. Use your linear algebra techniques to find the least squares regression line.

$$c_0 + c_1 x = y$$
$$c_0 + c_1(1) = 1$$
$$c_0 + c_2(2) = 2$$
$$c_0 + c_3(3) = 2$$

$$\rightarrow \quad A\vec{x} = \vec{b} \quad \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

The normal equations are $A^TA\vec{x} = A^T\vec{b}$

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

```
[A]T*[A]
                [3 6 ]
                [6 14]
[A]T*[B]
                  [5 ]
                  [11]
rref([C])
      [1 0 .6666666667]
      [0 1          .5]
Ans▶Frac
              [1 0 2/3]
              [0 1 1/2]
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and the solution is $x = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/2 \end{bmatrix}$

$\boxed{\text{The least squares regression line is } y = \frac{2}{3} + \frac{1}{2}x}$

8. (18 POINTS) Define the linear transformation T by $T(\mathbf{x}) = A\mathbf{x}$. Let

$$A = \begin{bmatrix} 5 & -3 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Find the following.

a. (8 POINTS) ker (T):

Ker(T): $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 5 & -3 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\boxed{\text{Ker}(T) = \{(0,0)\}}$

b. (1 POINT) nullity (T) = 0

leading ones

c. (8 POINTS) range (T):

$$A^T = \begin{bmatrix} 5 & 1 & 1 \\ -3 & 1 & -1 \end{bmatrix} \xrightarrow{\text{ref}} \begin{array}{ccc} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \\ \end{array} \begin{bmatrix} 1 & 0 & 1/4 \\ 0 & 1 & -1/4 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 0 & 1 \\ 0 & 4 & -1 \end{bmatrix}$$

$\vec{x}_3 = \frac{1}{4}x_1 - \frac{1}{4}x_2$

Let $x_1 = 4s$ and $x_2 = 4t$

$x_3 = s - t$

$\boxed{\text{range}(T) = \{(4s, 4t, s-t) : s,t \in \mathbb{R}\}}$

d. (1 POINT) rank (T) = 2

9. (10 POINTS) Let $T(1,1,1) = (2, 0, -1)$, $T(0, -1, 2) = (-3, 2, -1)$, and $T(1, 0, 1) = (1, 1, 0)$. Find the image of $T(0,2,-1)$.

$$(0,2,-1) = c_1(1,1,1) + c_2(0,-1,2) + c_3(1,0,1)$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & 2 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & -3/2 \end{bmatrix}$$

$$T((0,2,-1)) = T\left(\tfrac{3}{2}(1,1,1) - \tfrac{1}{2}(0,-1,2) - \tfrac{3}{2}(1,0,1)\right)$$

$$T(0,2,-1) = \tfrac{3}{2}T(1,1,1) - \tfrac{1}{2}T(0,-1,2) - \tfrac{3}{2}T(1,0,1)$$

$$T(0,2,-1) = \tfrac{3}{2}(2,0,-1) - \tfrac{1}{2}(-3,2,-1) - \tfrac{3}{2}(1,1,0) = \boxed{(3, -5/2, -1)}$$

10. (6 POINTS) Define the linear transformation $T : R^n \rightarrow R^m$ by $T(\mathbf{v}) = A\mathbf{v}$.

$$A = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix}. \rightarrow \begin{bmatrix} 1 & -2 & 0 & -7/2 & -4 \\ 0 & 0 & 1 & -1/2 & 0 \end{bmatrix}$$

a. $\dim(R^m) =$ 2

b. $\dim(R^n) =$ 5

11. (10 POINTS) Consider the linear transformation $T : R^2 \rightarrow R^2$, $T(x, y) = (2x + y, x - y)$.

a. (8 POINTS) Find ker(T).

Ker T: $T(\vec{x}) = \vec{0}$

$x = 2x + y = 0$

$y = x - y = 0$

$$\begin{aligned} 2x + y &= 0 \\ x - y &= 0 \\ \hline 3x &= 0 \\ x &= 0 \\ y &= 0 \end{aligned}$$

$\boxed{\text{So } \ker(T) = \{(0,0)\}}$

T is one-to-one

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b. (2 POINTS) T is (Circle ~~One~~ all that apply): one-to-one onto neither

rank$(T) = \dim(R^2) - \text{nullity}(T)$

$= 2 - 0$

$= 2$

so $\dim(A) = \text{rank}(A) = 2$

→ T is onto