MATH 270/GRACEY NAME______________________________

EXAM 2/Chapter 3-4.4

100 POINTS POSSIBLE

YOUR WORK MUST BE ORGANISED AND CLEAR

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1. (9 POINTS) Please circle true or false.

a. T F If A is an $n \times n$ matrix and c is a nonzero scalar, then the determinant of the matrix cA is given by $nc \cdot \det(A)$. $|cA| = c^n |A|$

b. T F If $\det(A) = 0$, then $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$. $\det(A) \neq 0$

c. T F In general, $\det(A+B) = \det(A) + \det(B)$.

2. (15 POINTS) Determine whether the set, together with the indicated operations, is a vector space. If it is not, then identify one of the ten vector space axioms that fail.

$M_{1,1}$ is a vector space

a. (10 POINTS) $M_{1,1}$ with the standard operations. $A = [a_{11}]$, $B = [b_{11}]$, $C = [c_{11}]$

c, d are scalars

1) $A + B = [a_{11} + b_{11}]$ ✓ closed under addition

2) $(A+B)+C = [a_{11}+b_{11}] + [c_{11}] = [a_{11}+b_{11}+c_{11}] = [a_{11}] + [b_{11}+c_{11}] = A + (B+C)$ ✓ associative under addition

3) $A + B = [a_{11}+b_{11}] = [b_{11}+a_{11}] = B + A$ ✓ comm under +

4) $A + \vec{0} = [a_{11}] + [0] = [a_{11}+0] = [a_{11}] = A$ ✓ add. identity

5) $A + (-A) = [a_{11}] + (-[a_{11}]) = [a_{11} - a_{11}] = [0] = \vec{0}$ ✓ add. inverse

6) $cA = c[a_{11}] = [ca_{11}]$ ✓ closed under scalar mult.

7) $(cd)A = [cda_{11}] = c[da_{11}] = c(dA)$ ✓ assoc. under mult

8) $c(A+B) = c[a_{11}+b_{11}] = [c(a_{11}+b_{11})] = [ca_{11}+cb_{11}] = [ca_{11}] + [cb_{11}] = c[a_{11}] + c[b_{11}] = cA + cB$ ✓ dist.

9) $(c+d)A = (c+d)[a_{11}] = [(c+d)a_{11}] = [ca_{11}+da_{11}] = [ca_{11}] + [da_{11}] = cA + dA$ ✓ dist.

10) $1A = 1[a_{11}] = [a_{11}] = A$ ✓ scalar identity

b. (5 POINTS) The set of integers with the standard operations.

The set of integers is not a vector space.

$2 \in$ set of integers

$\frac{1}{3}$ is a scalar

$\frac{1}{3} \cdot 2 = \frac{2}{3} \notin$ the set of integers.

3. (10 POINTS) Use a determinant to find the equation of the plane passing through $(3,1,-1)$, $(-2,4,7)$, and $(0,3,2)$.

$$\det\begin{bmatrix} x & y & z & 1 \\ 3 & 1 & -1 & 1 \\ -2 & 4 & 7 & 1 \\ 0 & 3 & 2 & 1 \end{bmatrix} = 0$$

$$0\begin{vmatrix} y & z & 1 \\ 1 & -1 & 1 \\ 4 & 7 & 1 \end{vmatrix} + 3\begin{vmatrix} x & z & 1 \\ 3 & -1 & 1 \\ -2 & 7 & 1 \end{vmatrix} - 2\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ -2 & 4 & 1 \end{vmatrix} + 1\begin{vmatrix} x & y & z \\ 3 & 1 & -1 \\ -2 & 4 & 7 \end{vmatrix} = 0$$

$$3[x(-8) - z(5) + 1(19)] - 2[x(-3) - y(5) + 1(14)] + 1[x(11) - y(19) + z(14)] = 0$$

$$-24x - 15z + 57 + 6x + 10y - 28 + 11x - 19y + 14z = 0$$

$$\boxed{-7x - 9y - z + 29 = 0}$$

4. (10 POINTS) If possible, write $\mathbf{v}$, as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

$$\mathbf{v} = (-2,6,6)$$
$$\mathbf{u}_1 = (1,2,-2)$$
$$\mathbf{u}_2 = (2,-1,1)$$

It is not possible to write $\vec{v}$ as a linear combination of $\vec{u}_1$ and $\vec{u}_2$.

$$c_1(1,2,-2) + c_2(2,-1,1) = (-2,6,6)$$

$$\begin{aligned} c_1 + 2c_2 &= -2 \\ 2c_1 - c_2 &= 6 \\ -2c_1 + c_2 &= 6 \end{aligned} \longrightarrow \begin{bmatrix} 1 & 2 & -2 \\ 2 & -1 & 6 \\ -2 & 1 & 6 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

inconsistent

5. (6 POINTS) Complete the following definition:

A nonempty subset W of a vector space V is called a subspace of V when W is

a vector space under the operations of addition and scalar multiplication defined in V.

6. (10 POINTS) Use Cramer's Rule to solve the linear system. **No graphing calculator—show all steps for full credit**.

$$5x - y = 3$$
$$2x + 3y = 7$$

$$x = \frac{\begin{vmatrix} 3 & -1 \\ 7 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & -1 \\ 2 & 3 \end{vmatrix}} = \frac{16}{17}$$

$$y = \frac{\begin{vmatrix} 5 & 3 \\ 2 & 7 \end{vmatrix}}{\begin{vmatrix} 5 & -1 \\ 2 & 3 \end{vmatrix}} = \frac{29}{17}$$

$$\boxed{\{(16/17, 29/17)\}}$$

7. (10 POINTS) Determine whether the set S spans R^3.

$S = \{(1,0,1),(1,1,0),(0,1,1)\}$

Let $\vec{u} = (u_1, u_2, u_3)$ be any vector in R^3.

$c_1(1,0,1) + c_2(1,1,0) + c_3(0,1,1) = (u_1, u_2, u_3)$

$$\begin{aligned} c_1 + c_2 \quad\quad &= u_1 \\ c_2 + c_3 &= u_2 \\ c_1 \quad\quad + c_3 &= u_3 \end{aligned}$$

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2 \neq 0$$ so there's a unique solution to the system.

$\boxed{\therefore S \text{ spans } R^3.}$

8. (20 POINTS) Consider the following matrices. **<u>Do not use a graphing calculator—show all steps for full credit</u>**.

$$A = \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} -3 & 4 \\ 6 & 1 \end{bmatrix}$$

a. (4 POINTS) Find $\det(A)$.

$\det(A) = -2 - 3 = \boxed{-5}$

b. (4 POINTS) Find $\det(B)$.

$\det(B) = -3 - 24 = \boxed{-27}$

c. (4 POINTS) Find $\det(A^{-1})$.

$$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{-5} = \boxed{-\frac{1}{5}}$$

d. (4 POINTS) Find $\det(A^T)$.

$\det(A^T) = \det(A) = \boxed{-5}$

e. (4 POINTS) Find $\det(AB)$

$$\begin{aligned} \det(AB) &= \det(A)\det(B) \\ &= (-5)(-27) \\ &= \boxed{135} \end{aligned}$$

9. (10 POINTS) Determine whether the set of vectors in $M_{2,2}$ is linearly independent or linearly dependent.

$$A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, B=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, C=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$$

$$c_1\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + c_2\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + c_3\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\begin{aligned}1c_1 + 0c_2 + 0c_3 &= 0\\ 0c_1 + 1c_2 + 0c_3 &= 0\\ 0c_1 + 0c_2 + 1c_3 &= 0\\ 1c_1 + 0c_2 + 0c_3 &= 0\end{aligned} \quad \longrightarrow \quad \begin{aligned}c_1 &= 0\\ c_2 &= 0\\ c_3 &= 0\\ c_1 &= 0\end{aligned}$$

$\boxed{c_1 = c_2 = c_3 = 0}$

so the set of vectors in $M_{2,2}$ is linearly independent.