MATH 270/GRACEY
EXAM 1/Chapter 1-2
100 POINTS POSSIBLE

NAME Key

YOUR WORK MUST BE ORGANIZED AND CLEAR
NO GRAPHING CALCULATOR IS PERMITTED

1. (9 POINTS) Find the solution set of the system of equations represented in the augmented matrix. If applicable, please parameterize.

$$\begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 3 & 9 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -7$

$3x_2 + 9x_3 = -2$

$x_2 = -3x_3 - \frac{2}{3}$

Let $x_3 = t$

$$\boxed{x_1 = -7,\ x_2 = -3t - \frac{2}{3},\ x_3 = t,\ t \in \mathbb{R}}$$

2. (15 POINTS) Please circle true or false.

a. T F A homogeneous system of equations must have at least one solution.

b. T F Multiplication of matrices is commutative.

c. T F If A and B are are $n \times n$ matrices and A is invertible then $\left(A^{-1}BA\right)^2 = AB^2A^{-1}$.

$(A^{-1}BA)(A^{-1}BA) = A^{-1}BI_nBA$

$= A^{-1}B^2A$

d. T F All $n \times n$ matrices are invertible.

e. T F If C is invertible, and $AC = BC$ then $A = B$.

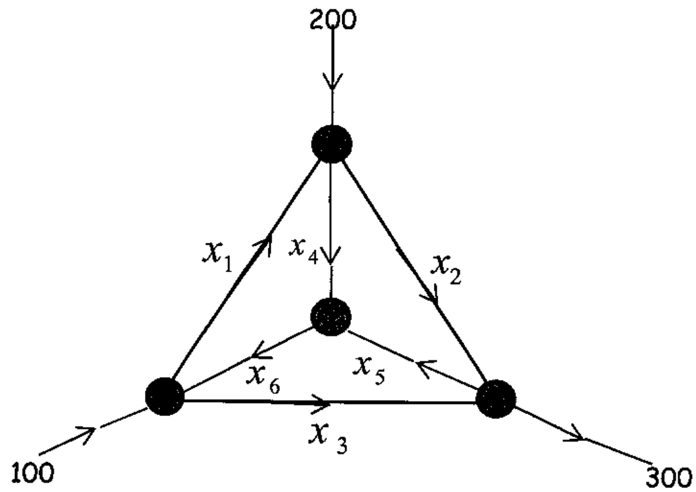
3. (9 POINTS) Please match the term/phrase with the correct definition. Each item is worth 3 points.

a. 3 b. 2 c. 1

Term/Phrase	Definition
a. SCALAR MULTIPLICATION	1. A matrix which does not have an inverse.
b. ELEMENTARY MATRIX	2. ...can be obtained from the identity matrix with a single row operation.
c. SINGULAR MATRIX	3. $c[a_{ij}]=[ca_{ij}]$

4. (10 POINTS) The figure shows the flow through a network.

Set up the system for x_i, $i=1,2,\ldots,6$



$x_1 + 200 = x_2 + x_4$

$x_2 + x_3 = x_5 + 300$

$x_4 + x_5 = x_6$

$x_6 + 100 = x_1 + x_3$

$$\boxed{\begin{aligned} x_1 - x_2 \quad - x_4 \quad &= -200 \\ x_2 + x_3 \quad - x_5 \quad &= 300 \\ x_4 + x_5 - x_6 &= 0 \\ -x_1 \quad - x_3 \quad + x_6 &= -100 \end{aligned}}$$

5. (10 POINTS) Use an *LU*-factorization of the coefficient matrix to solve the linear system. Recall, you need to solve the lower triangular system, $L\mathbf{y} = \mathbf{b}$, and the upper triangular system, $U\mathbf{x} = \mathbf{y}$.

$$\begin{aligned} x \quad\quad + z &= 3 \\ 2x + y + 2z &= 7 \\ 3x + 2y + 6z &= 8 \end{aligned}$$

$\boxed{(4, 1, -1) \quad \text{consistent system}}$

$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 6 \end{bmatrix}$

$\downarrow -2R_1 + R_2 \rightarrow R_2$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 2 & 6 \end{bmatrix}$

$\downarrow -3R_1 + R_3 \rightarrow R_3$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix}$

$\downarrow -2R_2 + R_3 \rightarrow R_3$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = U$

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$EA = U \rightarrow E_3E_2E_1A = U \rightarrow A = E_1^{-1}E_2^{-1}E_1^{-1}U$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = LU$

$L\vec{y} = \vec{b}$:

$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix}$

$y_1 = 3$

$2y_1 + y_2 = 7 \rightarrow y_2 = 1$

$3y_1 + 2y_2 + y_3 = 8 \rightarrow y_3 = -3$

$U\vec{x} = \vec{y}$:

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix}$

$x_1 + x_3 = 3 \rightarrow x_1 = 4$

$x_2 = 1$

$3x_3 = -3 \rightarrow x_3 = -1$

6. (18 POINTS) Consider the following matrix A.

$$A = \begin{bmatrix} -2 & 5 \\ -1 & 2 \end{bmatrix}$$

a. (10 POINTS) Find A^{-1}.

$$A^{-1}: \left[\begin{array}{cc|cc} -2 & 5 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right]$$

$$\downarrow R_1 + -2R_2 \to R_2$$

$$\left[\begin{array}{cc|cc} -2 & 5 & 1 & 0 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\downarrow R_1 + -5R_2 \to R_1$$

$$\left[\begin{array}{cc|cc} -2 & 0 & -4 & 10 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\downarrow \tfrac{-1}{2}R_1 \to R_1$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -5 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}}$$

b. (4 POINTS) Find $(A^T)^{-1}$.

$$(A^T)^{-1} = (A^{-1})^T = \boxed{\begin{bmatrix} 2 & 1 \\ -5 & -2 \end{bmatrix}}$$

c. (4 POINTS) Find $(5A)^{-1}$.

$$(5A)^{-1} = \frac{1}{5}A^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{2}{5} & -1 \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix}}$$

7. (9 POINTS) A country is divided into three regions. Each year, 10% of the residents of Region 1 move to Region 2 and 5% move to Region 3; 15% of the residents of Region 2 move to Region 1 and 5% move to Region 3; and 10% of the residents of Region 3 move to Region 1 and 10% move to Region 2. This year each region has a population of 100,000. Find the population of each region in (a) 1 year and (b) in 3 years.

$$\begin{array}{cc} \begin{array}{ccc} R_1 & R_2 & R_3 \end{array} & \\ \begin{bmatrix} 0.85 & 0.15 & 0.10 \\ 0.10 & 0.80 & 0.10 \\ 0.05 & 0.05 & 0.80 \end{bmatrix} & \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \end{array}$$

$$PX = \begin{bmatrix} 0.85 & 0.15 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.05 & 0.05 & 0.8 \end{bmatrix} \begin{bmatrix} 100000 \\ 100000 \\ 100000 \end{bmatrix} = \begin{bmatrix} 110000 \\ 100000 \\ 90000 \end{bmatrix} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \Bigg\} \text{In 1 year}$$

$$P(RX) = \begin{bmatrix} 0.85 & 0.15 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.05 & 0.05 & 0.8 \end{bmatrix} \begin{bmatrix} 110000 \\ 100000 \\ 90000 \end{bmatrix} = \begin{bmatrix} 117500 \\ 100000 \\ 82500 \end{bmatrix} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \Bigg\} \text{In 2 years}$$

$$P[P(PX)] = \begin{bmatrix} 0.85 & 0.15 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.05 & 0.05 & 0.8 \end{bmatrix} \begin{bmatrix} 117500 \\ 100000 \\ 82500 \end{bmatrix} = \begin{bmatrix} 123125 \\ 100000 \\ 76875 \end{bmatrix} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \Bigg\} \text{In 3 yrs}$$

8. (10 POINTS) A matrix is idempotent if $A^2 = A$. Prove that if A is an $n \times n$ matrix that is idempotent and invertible, then $A = I_n$.

Proof:

Let A be nxn, and Idempotent and Invertible.

$A = A^2$ (Idempotent)

$A = A \cdot A$

$A \cdot A^{-1} = A \cdot A \cdot A^{-1}$ (A is invertible)

$I_n = A I_n$

$I_n = A$

$A = I_n$ //

9. (10 POINTS) A square matrix is called skew-symmetric if $A^T = -A$. Prove that if A and B are $n \times n$ skew-symmetric matrices, then $A + B$ is skew-symmetric.

Proof:

Let A and B be skew-symmetric matrices and of size nxn.

$(A+B)^T = A^T + B^T$

$= (-A) + (-B)$

$= -(A+B)$. //