MATH 270/GRACEY NAME Key
EXAM 2/Chapter 3-4.7
100 POINTS POSSIBLE

YOUR WORK MUST BE ORGANIZED AND CLEAR

1. (8 POINTS) Please circle true or false.

a. T F If A is a 12 by 8 matrix of nullity 5, the number of independent vectors satisfying $A\mathbf{x} = \mathbf{0}$ is 7.

Skipped b. T F If the determinant of an $n \times n$ matrix A is 5, then the ~~matrix~~ determinant produced by interchanging two rows of A is also 5.

c. T F The columns of a matrix are the basis for the column space.

d. T F If a basis for a vector space has a dimension of n, then all bases of that vector space contain $n+1$ vectors.

2. (6 POINTS) Determine whether $W = \{(x, 4x) : x \text{ is a real number}\}$ is a subspace of R^2.

W is a nonempty subset of $\mathbb{R}^2$.

Closure under addition:

$\vec{u} = (u_1, 4u_1)$ and $\vec{v} = (v_1, 4v_1)$ are elements of W.

$\vec{u} + \vec{v} = (u_1, 4u_1) + (v_1, 4v_1)$

$= (u_1 + v_1, 4u_1 + 4v_1)$

$= (u_1 + v_1, 4(u_1 + v_1))$ ✓

Closure under scalar multiplication:

$\vec{u} = (u_1, 4u_1)$ and $c \in \mathbb{R}$.

$c\vec{u} = c(u_1, 4u_1)$

$= (cu_1, c(4u_1))$

$= (cu_1, 4(cu_1))$ ✓

$\boxed{\therefore \text{W is a subspace of } \mathbb{R}^2.}$

3. (6 POINTS) Use a determinant to find the equation of the plane passing through the points $(0,0,0)$, $(1,-1,0)$ and $(0,1,-1)$. Do not use your graphing calculator. Show all work for full credit.

$$\begin{vmatrix} x & y & z & 1 \\ \bar{0} & \overset{+}{0} & \bar{0} & \overset{+}{1} \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{vmatrix} = 0$$

$$1\begin{vmatrix} \overset{+}{x} & \overset{-}{y} & \overset{+}{z} \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0$$

$$x\begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix} - y\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + z\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 0$$

$$x(1-0) - y(-1-0) + z(1-0) = 0$$

$$\boxed{x + y + z = 0}$$

4. (6 POINTS) Find the coordinate matrix of $\mathbf{x} = (3,-3,0)$ relative to the nonstandard basis $B' = \{(1,2,3),(1,2,0),(0,-6,2)\}$.

$$c_1(1,2,3) + c_2(1,2,0) + c_3(0,-6,2) = (3,-3,0)$$

$$\begin{aligned} c_1 + c_2 \quad\quad &= 3 \\ 2c_1 + 2c_2 - 6c_3 &= -3 \\ 3c_1 \quad\quad + 2c_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 2 & 2 & -6 & -3 \\ 3 & 0 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1.5 \end{array}\right]$$

$c_1 = -1$

$c_2 = 4$

$c_3 = \frac{3}{2}$

so, $\boxed{[\vec{x}]_{B'} = \begin{bmatrix} -1 \\ 4 \\ 3/2 \end{bmatrix}}$

5. (6 POINTS) Let $S = \{\mathbf{u}, \mathbf{v}\}$ be a linearly independent set. Prove that the set $S_1 = \{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}\}$ is also linearly independent.

PF: Let c_i, $i = 1,2 \in \mathbb{R}$.

$$c_1(\vec{u}+\vec{v}) + c_2(\vec{u}-\vec{v}) = \vec{0}$$
$$(c_1\vec{u} + c_1\vec{v}) + (c_2\vec{u} - c_2\vec{v}) = \vec{0}$$
$$(c_1\vec{u} + c_2\vec{u}) + (c_1\vec{v} - c_2\vec{v} = \vec{0}$$
$$(c_1+c_2)\vec{u} + (c_1-c_2)\vec{v} = \vec{0} \Rightarrow c_1+c_2=0 \text{ and } c_1-c_2=0$$

[S is linearly independent]

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 - c_2 &= 0 \\ \hline 2c_1 &= 0 \\ c_1 &= 0 \rightarrow c_2 = 0. \end{aligned}$$

$\therefore$ S_1 is also linearly independent. //

6. (8 POINTS) Determine whether $S = \{(1,5,3),(0,1,2),(0,0,6)\}$ is a basis for R^3.

$$c_1(1,5,3) + c_2(0,1,2) + c_3(0,0,6) = (0,0,0)$$

$$\begin{aligned} c_1 &= 0 \\ 5c_1 + c_2 &= 0 \rightarrow 5(0) + c_2 = 0 \rightarrow c_2 = 0 \\ 3c_1 + 2c_2 + 6c_3 &= 0 \rightarrow 3(0) + 2(0) + 6c_3 = 0 \rightarrow 6c_3 = 0 \rightarrow c_3 = 0 \end{aligned}$$

Since $c_1 = c_2 = c_3 = 0$, S is linearly independent. Furthermore, since the dimension of R^3 is 3 and S contains 3 linearly independent vectors, S is a basis for R^3.

7. (8 POINTS/2 POINTS EACH) Suppose the determinants of two matrices of order $n \times n$ are:

$$|A| = -3 \, , |B| = 7$$

a. Find $\det\left(A^{-1}\right)$.

$\det(A^{-1}) = \dfrac{1}{\det(A)} = \boxed{-\frac{1}{3}}$

b. Find $\det\left(A^{T}\right)$.

$\det(A^{T}) = \det(A) = \boxed{-3}$

c. Find $\det(AB)$.

$\det(AB) = \det(A)\det(B) = (-3)(7) \to = \boxed{-21}$

d. $\det(5B)^{-1}$.

$\det(5B)^{-1} = \frac{1}{5}\det B^{-1} = \frac{1}{5} \cdot \frac{1}{7} = \boxed{\frac{1}{35}}$

8. (8 POINTS) Write the solution of the consistent system below in the form $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$, where $\mathbf{x}_h$ is a solution of $A\mathbf{x} = \mathbf{0}$ and $\mathbf{x}_p$ is a particular solution of $A\mathbf{x} = \mathbf{b}$.

$$\begin{aligned} x + 3y + 10z &= 18 \\ -2x + 7y + 32z &= 29 \\ -x + 3y + 14z &= 12 \\ x + \ y + \ 2z &= 8 \end{aligned}$$

To find $\vec{x}_h$:

$$\left[\begin{array}{ccc|c} 1 & 3 & 10 & 0 \\ -2 & 7 & 32 & 0 \\ -1 & 3 & 14 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{array}{l} x \quad -2z = 0 \to x = 2t \\ y \ +4z = 0 \to y = -4t \\ \qquad\qquad z = t \end{array}$$

$$\vec{x}_h = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -4t \\ t \end{bmatrix}$$

To find $\vec{x}_p$:

$$\left[\begin{array}{ccc|c} 1 & 3 & 10 & 18 \\ -2 & 7 & 32 & 29 \\ -1 & 3 & 14 & 12 \\ 1 & 1 & 2 & 8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{array}{l} x \quad -2z = 3 \to x = 3+2s \\ y \ +4z = 5 \to y = 5-4s \\ \qquad\qquad z = s \end{array}$$

$$\vec{x}_p = \begin{bmatrix} 3+2s \\ 5-4s \\ s \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} \to \text{let } s = 0$$

$$\vec{x} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2t \\ -4t \\ t \end{bmatrix} \to \boxed{\vec{x} = \{(3,5,0) + (2t,-4t,t) : t \in \mathbb{R}\} \text{ is a solution}}$$

9. (20 POINTS) Consider the matrix

$$A=\begin{bmatrix} 1 & 3 & -2 & 4 \\ 0 & 1 & -1 & 2 \\ -2 & -6 & 4 & -8 \end{bmatrix}.$$

Please find the following. You may use your graphing calculator.

a. (6 POINTS) A basis for the row space of A:

b. (6 POINTS) A basis for the column space of A:

c. (6 POINTS) A basis for the nullspace of A:

d. (1 POINT) $\operatorname{rank}(A)$ = ________

e. (1 POINT) $\operatorname{nullity}(A)$ = _______

10. (8 POINTS)

a. (6 POINTS) Find a basis for the vector space of all lower triangular 3×3 matrices.

b. (2 POINTS) What is the dimension of this space?

11. (6 POINTS) Determine whether $S=\{x^2+7, 3x+5\}$ is a basis for P_2.

12. (10 POINTS, 5 POINTS EACH) Write down the definitions and theorems exactly as given in the text.

a. **DEFINITION OF LINEAR DEPENDENCE AND LINEAR INDEPENDENCE**

b. **THEOREM 4.11: NUMBER OF VECTORS IN A BASIS**