Key

Part I: (60 Points/10 Points each) Problems 1-7: Ascertain whether the infinite series converges or diverges. You must include the test, show how the condition(s) are met, run the test, and provide a conclusion. **Please complete 6 out of the 7 problems**. Be sure to write down your evil plan(s) or strategies; especially if you get stuck on a problem. **Cross out the problem that you do not want graded**.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

Step 1: Identify the test(s) and conditions (if applicable).

Ratio test. $a_n = \frac{(2n)!}{n^5}$ is a series with nonzero terms.

Step 2: Run the test.

$$\lim_{n\to\infty} \left| \frac{[2(n+1)]!}{(n+1)^5} \cdot \frac{n^5}{(2n)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(2n+2)!\, n^5}{(n+1)^5 (2n)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)\cancel{(2n)!}\, n^5}{(n+1)^5 \cancel{(2n)!}} \right| = \infty$$

Step 3: Conclusion.

$\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{n^5}$ diverges by the Ratio test.

2. $\sum_{n=4}^{\infty} \frac{(-1)^n n}{n-3}$

Step 1: Identify the test(s) and conditions (if applicable).

Alternating series test.

$a_n = \frac{n}{n-3}$

Step 2: Run the test.

$\lim_{n \to \infty} \frac{n}{n-3} = 1$

Step 3: Conclusion.

$\sum_{n=4}^{\infty} \frac{(-1)^n}{n-3}$ diverges by the nth term test for divergence.

3. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+3n}}$

Step 1: Identify the test(s) and conditions (if applicable).

Limit comparison test.

$a_n = \frac{n}{\sqrt{n^3+3n}} > 0, \quad b_n = \frac{n}{\sqrt{n^3}} = \frac{n}{n^{3/2}} = \frac{1}{n^{1/2}} > 0$

Step 2: Run the test.

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^3+3n}} \cdot \frac{n^{1/2}}{1}$$

$$= \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3+3n}}$$

$= 1$ which is finite & positive.

$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series $[p = 1/2]$.

Step 3: Conclusion.

$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+3n}}$ diverges by the limit comparison test.

4. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Step 1: Identify the test(s) and conditions (if applicable).

Integral Test. Let $f(x) = \frac{\ln x}{x}$. f is continuous on $[1, \infty)$, and positive on $[2, \infty)$.

$$f'(x) = \frac{\left(\frac{1}{x}\right)x - (\ln x)(1)}{x^2}$$

so $f'(x) > 0$ on $[3, \infty)$.

$$f'(x) = \frac{1 - \ln x}{x^2}$$

$$1 - \ln x = 0$$

$$1 = \ln x$$

$$x = e^1 = e$$

Step 2: Run the test.

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{b \to \infty} \int_1^b \frac{\ln x}{x}\,dx$$

$$= \lim_{b \to \infty} \frac{1}{2}(\ln x)^2 \Big|_{x=1}^{x=b}$$

$$= \frac{1}{2} \lim_{b \to \infty} \left[(\ln b)^2 - (\ln 1)^2\right]$$

$$= \frac{1}{2}(\infty - 0)$$

$$= \infty \rightarrow \text{diverges}$$

Step 3: Conclusion.

$\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by the integral test.

5. $\sum_{n=1}^{\infty}\left(\frac{4n}{7n-1}\right)^{n}$

Step 1: Identify the test(s) and conditions (if applicable).

Root test.

$\sum_{n=1}^{\infty}\left(\frac{4n}{7n-1}\right)^{n}$ is a series.

Step 2: Run the test.

$$\lim_{n\to\infty}\left|\sqrt[n]{\left(\frac{4n}{7n-1}\right)^{n}}\right| = \lim_{n\to\infty}\left|\frac{4n}{7n-1}\right| = \frac{4}{7} < 1$$

Step 3: Conclusion.

$\sum_{n=1}^{\infty}\left(\frac{4n}{7n-1}\right)^{n}$ converges by the root test.

6. $\sum_{n=1}^{\infty}\left(\sqrt{e}\right)^{n}$

Step 1: Identify the test(s) and conditions (if applicable).

$\sqrt{e} \approx 1.65$

Step 2: Run the test.

N/A

Step 3: Conclusion.

$\sum_{n=1}^{\infty}(\sqrt{e})^n$ is a divergent geometric series $[|r| = |\sqrt{e}| \approx 1.65]$.

7. $\sum_{n=1}^{\infty} \frac{2^n}{3^n - 1}$

DCT won't work

$b_n = \frac{2^n}{3^n}$ $\quad a_n = \frac{2^n}{3^n - 1}$

$a_1 = \frac{2}{2} = 1 > b_1 = \frac{2}{3}$

$a_2 = \frac{4}{8} > b_2 = \frac{4}{9}$

Step 1: Identify the test(s) and conditions (if applicable).

Limit comparison test

$a_n = \frac{2^n}{3^n - 1} > 0$

$b_n = \frac{2^n}{3^n} > 0$

Step 2: Run the test.

$$\lim_{n\to\infty} \left[\frac{2^n}{3^n - 1} \cdot \frac{3^n}{2^n} \right]$$

$$= \lim_{n\to\infty} \frac{3^n}{3^n - 1}$$

$$\overset{LR}{=} \lim_{n\to\infty} \frac{(\ln 3) \cdot 3^n}{(\ln 3) 3^n}$$

$= 1$ which is finite and positive.

$\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent geometric series $\left[|r| = \left|\frac{2}{3}\right| < 1 \right]$.

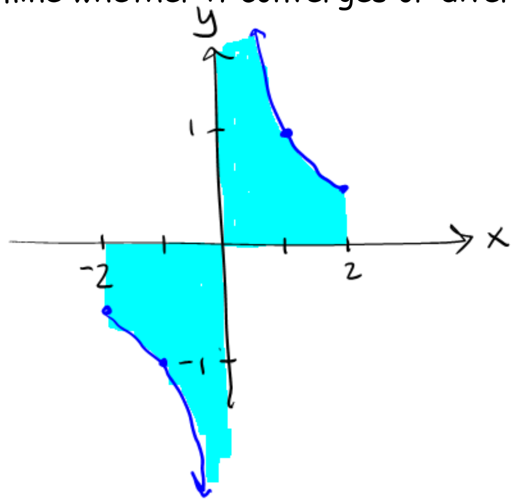
Step 3: Conclusion.

$\sum_{n=1}^{\infty} \frac{2^n}{3^n - 1}$ converges by the limit comparison test.

Part II: (30 points/10 points each) Problems 8-10. Complete the following problems.

8. Evaluate the definite integral and determine whether it converges or diverges.

consider $\int_{-2}^{2} \frac{1}{x} dx = \int_{-2}^{0} \frac{dx}{x} + \int_{0}^{2} \frac{dx}{x}$



$\int_{0}^{2} \frac{dx}{x}$

$= \lim_{a \to 0^+} \int_{a}^{2} \frac{dx}{x}$

$= \lim_{a \to 0^+} \left(\ln|x| \Big|_{x=a}^{x=2} \right)$

$= \lim_{a \to 0^+} \left[\ln|2| - \ln|a| \right]$

$= \left[\ln 2 - \infty \right]$

$= -\infty$

$\boxed{\text{So } \int_{-2}^{2} \frac{dx}{x} \text{ diverges.}}$

9. Find the sum of the convergent series.

$$\sum_{n=1}^{\infty}\left[\left(\frac{4}{5}\right)^n - \frac{1}{(n+1)(n+2)}\right]$$

$$= \sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n - \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}$$

$$= 4 - \frac{1}{2}$$

$$= \boxed{\frac{7}{2}}$$

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$1 = An + 2A + Bn + B$$

$$0n + 1 = (A+B)n + (2A+B)$$

$$A+B=0 \rightarrow A=-B$$

$$2A+B=1 \rightarrow 2(-B)+B=1$$

$$-B=1$$

$$B=-1$$

$$A=-(-1)=1$$

$$\sum_{n=1}^{\infty}\left[\frac{1}{n+1} - \frac{1}{n+2}\right]$$

$$= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right) + \cdots$$

$$= \frac{1}{2} + \lim_{n\to\infty} a_{n+1}$$

$$= \frac{1}{2} + \lim_{n\to\infty}\left[\frac{1}{n+1+1} - \frac{1}{n+1+2}\right]$$

$$= \frac{1}{2} + \lim_{n\to\infty}\left[\frac{1}{n+2} - \frac{1}{n+3}\right]$$

$$= \frac{1}{2} + 0$$

$$= \frac{1}{2}$$

$$\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n = \sum_{n=0}^{\infty}\left(\frac{4}{5}\right)^n - \left(\frac{4}{5}\right)^0$$

$$= \frac{1}{1-\frac{4}{5}} - 1$$

$$= \frac{1}{\frac{1}{5}} - 1$$

$$= 4$$

10. Determine whether the series converges absolutely or conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} = \frac{\cos \pi}{\sqrt{1}} + \frac{\cos 2\pi}{\sqrt{2}} + \frac{\cos 3\pi}{\sqrt{3}} + \frac{\cos 4\pi}{\sqrt{4}} + \cdots$$

$$= \frac{-1}{\sqrt{1}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

$\sum_{n=1}^{\infty} \left| \frac{\cos n\pi}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ which is a divergent p-series $[p = \frac{1}{2}]$.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the AST 1) $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$ ✓ 2) $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ for all n.

$\boxed{\text{So } \sum_{n=1}^{\infty} \frac{\cos n\pi}{\sqrt{n}} \text{ is conditionally convergent.}}$

Part II: (10 points/2 points each) Problems 11-15. True or False.

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11. T **F** If $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n$ converges.

12. T **F** If $0 < a_n \le b_n$, and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

13. **T** F If $\{a_n\}$ is bounded and monotonic, $\{a_n\}$ converges.

14. T **F** The *n*th Term Test may be used to show convergence.

15. **T** F If $\sum_{n=1}^{\infty} a_n$ converges and has a sum of 3 and $\sum_{n=1}^{\infty} b_n$ converges and has a sum of 5, $\sum_{n=1}^{\infty} (a_n + b_n)$ will also converge and have a sum of 8.