Part I: (60 Points/10 Points each) Problems 1-7: Evaluate the definite integrals and find the indefinite integrals. **Please complete 6 out of the 7 problems**. Be sure to write down your evil plan(s) or strategies; especially if you get stuck on a problem. Provide exact answers only. **Cross out the problem that you do not want graded**.

1. $\int \arcsin 2x\,dx = (\arcsin 2x)(x) - \int (x)\left(\frac{2\,dx}{\sqrt{1-4x^2}}\right)$

$= x\arcsin 2x - 2\left(-\frac{1}{8}\right)\int x\,(1-4x^2)^{-1/2}\,dx\,(-8)$

$= x\arcsin 2x + \frac{1}{4}\frac{(1-4x^2)^{1/2}}{1/2} + C$

$= \boxed{x\arcsin 2x + \frac{1}{2}\sqrt{1-4x^2} + C}$

IBP

$u = \arcsin 2x$

$\frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}$

$du = \frac{2\,dx}{\sqrt{1-4x^2}}$

$\int dv = \int dx$

$v = x$

$g(x) = 1-4x^2$

$g'(x) = -8x$

$\int f(g(x))g'(x)\,dx$

$= F(g(x)) + C$

2. $\int \left(\frac{\ln x}{\sqrt{x}}\right)^2 dx = \int \frac{(\ln x)^2}{x} dx$

$= \int \frac{(u)^2}{\cancel{x}} \cdot (\cancel{x} du)$

$= \int u^2 du$

$= \frac{1}{3} u^3 + C$

$= \boxed{\frac{1}{3} (\ln x)^3 + C}$

u-sub

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x du$

3. $\int_0^2 (x-1)\sqrt{2x+1}\,dx = \int_1^5 \left[\left(\frac{1}{2}(u-1)\right)-1\right](u^{1/2})\left(\frac{du}{2}\right)$

$= \frac{1}{2}\int_1^5 \left(\frac{1}{2}u - \frac{3}{2}\right)u^{1/2}\,du$

$= \frac{1}{4}\int_1^5 (u-3)u^{1/2}\,du$

$= \frac{1}{4}\int_1^5 (u^{3/2} - 3u^{1/2})\,du$

$= \frac{1}{4}\left[\frac{2}{5}u^{5/2} - 2u^{3/2}\right]_{u=1}^{u=5}$

$= \frac{1}{4}\left[\left(\frac{2}{5}(\sqrt{5})^5\right) - 2\left((\sqrt{5})^3\right) - \left(\frac{2}{5}\cdot 1 - 2\cdot 1\right)\right]$

$= \frac{1}{4}\left(\frac{2}{\cancel{5}}\,\overset{5}{\cancel{25}}\sqrt{5} - 2\cdot 5\sqrt{5} - \frac{2}{5} + 2\right)$

$= \frac{1}{4}\left(10\sqrt{5} - 10\sqrt{5} + \frac{8}{5}\right)$

$= \frac{1}{4}\cdot\frac{8}{5}$

$= \boxed{\frac{2}{5}}$

IBP or u-sub

u-sub:

$u = 2x+1 \rightarrow x = \frac{1}{2}(u-1)$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

upper limit:

$u(2) = 2(2)+1 = 5$

lower limit:

$u(0) = 2(0)+1) = 1$

IBP: $u = x-1$ $\quad \int dv = \int (2x+1)^{1/2}dx$

$\frac{du}{dx} = 1$ $\quad v = \frac{1}{2}\frac{(2x+1)^{3/2}}{3/2}$

$du = dx$ $\quad v = \frac{1}{3}(2x+1)^{3/2}$

IBP

$\int (x-1)(2x+1)^{1/2}\,dx = (x-1)\left[\frac{1}{3}(2x+1)^{3/2}\right] - \int \left[\frac{1}{3}(2x+1)^{3/2}\right](dx)$

$= \frac{1}{3}(x-1)(2x+1)^{3/2} - \frac{1}{3}\cdot\frac{1}{2}\int (2x+1)^{3/2}\,dx\cdot 2$

$= \frac{1}{3}(x-1)(2x+1)^{3/2} - \frac{1}{6}\left(\frac{(2x+1)^{5/2}}{5/2}\right) + C$

$= \frac{1}{3}(x-1)(2x+1)^{3/2} - \frac{1}{15}(2x+1)^{5/2} + C$

$g(x) = 2x+1$

$g'(x) = 2$

$\int f(g(x))g'(x)\,dx$

$= F(g(x)) + C$

So $\int_0^2 (x-1)(2x+1)^{1/2}\,dx = \left[\left(\frac{1}{3}(2-1)(2\cdot 2+1)^{3/2} - \frac{1}{15}(2\cdot 2+1)^{5/2}\right] - \left[\frac{1}{3}(0-1)(2\cdot 0+1)^{3/2} - \frac{1}{15}(2\cdot 0+1)^{5/2}\right]$

$= \left[\frac{1}{3}(\sqrt{5})^3 - \frac{1}{15}(\sqrt{5})^5 - \left(-\frac{1}{3}\right)(1)^{3/2} + \frac{1}{15}(1)^{5/2}\right]$

$= \frac{1}{3}\cdot 5\sqrt{5} - \frac{1}{\cancel{15}_3}\cdot \overset{5}{\cancel{25}}\sqrt{5} + \frac{1}{3} + \frac{1}{15}$

$$= \frac{5}{3}\sqrt{5} - \frac{5}{3}\sqrt{5} + \frac{6}{15}$$

$$= \boxed{\frac{2}{5}}$$

4. $\int \tan^4\theta \sec^4\theta\, d\theta = \int (\tan\theta)^4 (\sec^2\theta)(\sec^2\theta)\, d\theta$

$= \int (\tan\theta)^4 \left[(\tan\theta)^2 + 1\right] \sec^2\theta\, d\theta$

$= \int \left[(\tan\theta)^6 + (\tan\theta)^4\right] \sec^2\theta\, d\theta$

$= \boxed{\frac{\tan^7\theta}{7} + \frac{\tan^5\theta}{5} + C}$

$g(\theta) = \tan\theta$

$g'(\theta) = \sec^2\theta$

$\int f(g(\theta))g'(\theta)\, d\theta$

$= F(g(\theta)) + C$

5. $\int \sin 5x \cos x dx = \frac{1}{2}\int \left(\sin(5-1)x + \sin(5+1)x\right)dx$

$$= \frac{1}{2}\int \left(\sin 4x + \sin 6x\right)dx$$

$$= \frac{1}{2}\left[\left(-\frac{\cos 4x}{4}\right) + \left(-\frac{\cos 6x}{6}\right)\right] + C$$

$$\boxed{= -\frac{1}{8}\cos 4x - \frac{1}{12}\cos 6x + C}$$

or

$$\boxed{= -\frac{1}{24}\left(3\cos 4x - 2\cos 6x\right) + C}$$

$a^2 = 1 \rightarrow a = 1$

6. $\int_0^1 \frac{1}{x^2+1} dx = \frac{1}{1} \arctan \frac{x}{1} \Big]_{x=0}^{x=1}$

$= \arctan 1 - \arctan 0$

$= \frac{\pi}{4} - 0$

$= \boxed{\frac{\pi}{4}}$

7. $\int_0^1 \left[\frac{x^2 - x}{x^2 + x + 1}\right] dx$

$$= \int_0^1 \left(1 - \frac{2x+1}{x^2+x+1} \right) dx$$

Long Division

$$\begin{array}{r} 1 - \frac{2x+1}{x^2+x+1} \\ (x^2+x+1)\overline{)\,x^2 - x + 0} \\ -(x^2+x+1) \\ \hline -2x-1 \end{array}$$

$g(x) = x^2 + x + 1$

$g'(x) = 2x + 1$

$$= \Big[x - \ln|x^2+x+1| \Big]_{x=0}^{x=1}$$

$$= \Big[(1 - \ln|1^2+1+1|) - (0 - \ln|0^2+0+1|) \Big]$$

$$= 1 - \ln 3 + \ln 1$$

$$= 1 - \ln 3 + 0$$

$$= \boxed{1 - \ln 3}$$

Part II: (14 Points) Problems 8-9: Find the indefinite integrals. **Please complete 1 out of the 2 problems**. Be sure to write down your evil plan(s) or strategies; especially if you get stuck on a problem. Provide exact answers only. **Cross out the problem that you do not want graded**.

8. $\int e^{-x} \sin x dx = (\sin x)(-e^{-x}) - \int (-e^{-x})(\cos x \, dx)$

IBP 1:

$u = \sin x \qquad \int dv = \int e^{-x} dx$

$\frac{du}{dx} = \cos x \qquad v = -e^{-x}$

$du = \cos x dx$

$\int e^{-x} \sin x dx = -e^{-x} \sin x + \int e^{-x} \cos x dx$

$\int e^{-x} \sin x dx = -e^{-x} \sin x + \left[(\cos x)(-e^{-x}) - \int (-e^{-x})(-\sin x dx)\right]$

IBP 2:

$u = \cos x \qquad \int dv = \int e^{-x} dx$

$\frac{du}{dx} = -\sin x \qquad v = -e^{-x}$

$du = -\sin x dx$

$\int e^{-x} \sin x dx = -e^{-x} \sin x - e^{-x} \cos x - \cancel{\int e^{-x} \sin x dx}$

$+ e^{-x} \sin x dx \qquad\qquad + \cancel{\int e^{-x} \sin x dx}$

$2\int e^{-x} \sin x dx = -e^{-x}(\sin x + \cos x) + C$

$$\boxed{\int e^{-x} \sin x dx = -\frac{1}{2} e^{-x}(\sin x + \cos x) + C}$$

9. $\int \sec^3 x\,dx = \int \sec x \sec^2 x\,dx$

IBP:

$u = \sec x$ $\qquad \int dv = \int \sec^2 x\,dx$

$\frac{du}{dx} = \sec x \tan x$ $\qquad v = \tan x$

$du = \sec x \tan x\,dx$

$$\int \sec^3 x\,dx = (\sec x)(\tan x) - \int (\tan x)(\sec x \tan x\,dx)$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \tan^2 x \sec x\,dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int (\sec^3 x - \sec x)\,dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \cancel{\int \sec^3 x\,dx} + \int \sec x\,dx$$

$$+\int \sec^3 x\,dx \qquad\qquad + \cancel{\int \sec^3 x\,dx}$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\boxed{\int \sec^3 x\,dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C}$$

Part III: (16 Points/8 points each). Problems 10-11. Evaluate the following limits. Exact answers only, please.

10. $\lim\limits_{x\to\infty} \frac{\ln x}{x} \overset{D.S.}{=} \frac{\infty}{\infty}$ indeterminate → in correct form to use L'Hôpital's rule

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$= \frac{0}{1}$$

$$= \boxed{0}$$

11. $\lim\limits_{x\to 0^+} x^{\sqrt{x}} \overset{D.S.}{=} 0^0$ indeterminate not in right format for L'Hôpital's Rule

$$x^{\sqrt{x}} = e^{\ln x^{\sqrt{x}}}$$

$$= e^{\sqrt{x}\ln x}$$

now $\lim\limits_{x\to 0^+} \sqrt{x}\ln x \overset{D.S.}{=} 0\cdot(-\infty)$ indeterminate but not in right format

$$= e^{\frac{\ln x}{\frac{1}{\sqrt{x}}}}$$

now $\lim\limits_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} = \frac{-\infty}{\infty}$

now in correct form for L'Hôpital's rule

$$\lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} e^{\ln x/(1/\sqrt{x})}$$

$$= e^{\lim_{x\to 0^+} \ln x / \frac{1}{\sqrt{x}}}$$

$$= e^{\lim_{x\to 0^+} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}\frac{1}{\sqrt{x}}}}$$

$$\frac{d}{dx} x^{-1/2} = -\frac{1}{2} x^{-3/2}$$

$$= e^{\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{2x^{-3/2}}}}$$

$$= e^{-\frac{2x^{-3/2}}{x}}$$

$$= e^{-2x^{-1/2}}$$

$$= e^{-\frac{2}{\sqrt{x}}}$$

$$= e^{-0}$$

$$= \boxed{1}$$

Part V: (10 Points). Problem 12. Solve the following application. Exact answers only, please.

Find the area of the region bounded by $f(x) = \cos^4 x$, $y = 0$, $x = \dfrac{\pi}{4}$, and. $x = \dfrac{\pi}{3}$.

$$A = \int_{\pi/4}^{\pi/3} (\cos x)^4 \, dx$$

$$A = \int_{\pi/4}^{\pi/3} \left[(\cos x)^2\right]^2 dx$$

$$A = \int_{\pi/4}^{\pi/3} \left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

$$A = \frac{1}{4} \int_{\pi/4}^{\pi/3} \left(1 + 2\cos 2x + (\cos 2x)^2\right) dx$$

$$A = \frac{1}{4}\left[\int_{\pi/4}^{\pi/3} (1 + 2\cos 2x \, dx) + \frac{1}{2} \int_{\pi/4}^{\pi/3} (1 + \cos 4x)\, dx\right]$$

$$A = \frac{1}{4}\Big[x + \sin 2x\Big]_{x=\pi/4}^{x=\pi/3} + \frac{1}{8}\left[x + \frac{\sin 4x}{4}\right]_{x=\pi/4}^{x=\pi/3}$$

$$A = \frac{1}{4}\left[\left(\pi/3 + \sin\frac{2\pi}{3}\right) - \left(\frac{\pi}{4} + \sin\frac{\pi}{2}\right)\right] + \frac{1}{8}\left[\left(\frac{\pi}{3} + \frac{1}{4}\sin\frac{4\pi}{3}\right) - \left(\frac{\pi}{4} + \frac{1}{4}\sin\pi\right)\right]$$

$$A = \frac{1}{4}\left[\frac{4\pi - 3\pi}{12} + \frac{\sqrt{3}}{2} - 1\right] + \frac{1}{8}\left[\frac{4\pi - 3\pi}{12} + \frac{1}{4}\left(-\frac{\sqrt{3}}{2}\right) - \frac{1}{4}\cdot 0\right]$$

$$A = \frac{1}{4}\cdot\frac{\pi}{12} + \frac{\sqrt{3}}{8} - \frac{1}{4} + \frac{1}{8}\cdot\frac{\pi}{12} - \frac{\sqrt{3}}{64}$$

$$A = \frac{2\pi + \pi}{96} + \frac{8\sqrt{3} - \sqrt{3}}{64} - \frac{1}{4}$$

$$A = \frac{\pi}{96} + \frac{7\sqrt{3}}{64} - \frac{1}{4}$$

$$\boxed{A = \frac{1}{192}\left(2\pi + 21\sqrt{3} - 24\right) \text{ sq units}}$$

$$\sin mx \sin nx = \frac{1}{2}(\cos[(m-n)x] - \cos[(m+n)x])$$

$$\sin mx \cos nx = \frac{1}{2}(\sin[(m-n)x] + \sin[(m+n)x])$$

$$\cos mx \cos nx = \frac{1}{2}(\cos[(m-n)x] + \cos[(m+n)x])$$