NAME Key

Part I: (30 Points) Problems 1-4: Complete the following problems.

1. (10 POINTS)

   a. (8 POINTS) Find the 2nd degree Taylor polynomial for $f(x) = x^2 \sin x$, centered at $\frac{\pi}{2}$.

$f(x) = x^2 \sin x$

$f'(x) = 2x \sin x + x^2 \cos x$

$f''(x) = 2\sin x + 2x\cos x + 2x\cos x - x^2 \sin x$

$f''(x) = 2\sin x + 4x\cos x - x^2 \sin x$

$f(\pi/2) = \frac{\pi^2}{4}$

$f'(\pi/2) = \pi$

$f''(\pi/2) = 2 - \frac{\pi^2}{4} = \frac{8-\pi^2}{4}$

$P_2(x) = \frac{\pi^2}{4} + \pi\left(x - \frac{\pi}{2}\right) + \frac{8-\pi^2}{2!\,4}\left(x - \frac{\pi}{2}\right)^2$

$$\boxed{P_2(x) = \frac{\pi^2}{4} + \pi\left(x - \frac{\pi}{2}\right) + \frac{8-\pi^2}{8}\left(x - \frac{\pi}{2}\right)^2}$$

   b. (2 POINTS) Use your result from part a to approximate $f\left(\frac{3\pi}{8}\right)$

$f\left(\frac{3\pi}{8}\right) \approx P_2\left(\frac{3\pi}{8}\right)$

$= \frac{\pi^2}{4} + \pi\left(\frac{3\pi}{8} - \frac{\pi}{2}\right) + \frac{8-\pi^2}{8}\left(\frac{3\pi}{8} - \frac{\pi}{2}\right)^2$

$= \boxed{1.1977}$

2. (10 POINTS)

a. (8 POINTS) Find the 4th degree Maclaurin polynomial for $f(x)=\dfrac{1}{x+1}$.

$f(x) = (x+1)^{-1}$ $\qquad f(0) = 1$

$f'(x) = -(x+1)^{-2}$ $\qquad f'(0) = -1$

$f''(x) = 2(x+1)^{-3}$ $\qquad f''(0) = 2$

$f'''(x) = -6(x+1)^{-4}$ $\qquad f'''(0) = -6$

$f^{(4)}(x) = 24(x+1)^{-5}$ $\qquad f^{(4)}(0) = 24$

$$P_4(x) = 1 - x + \frac{2x^2}{2!} - \frac{6x^3}{3!} + \frac{24x^4}{4!}$$

$$\boxed{P_4(x) = 1 - x + x^2 - x^3 + x^4}$$

b. (2 POINTS) Use your result from part a to approximate $f\left(\dfrac{1}{4}\right)$

$f\left(\frac{1}{4}\right) \approx P_4(x)$

$= 1 - \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4$

$= \boxed{0.8008}$

3. (6 POINTS) Find the radius of convergence and the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!(x-5)^n}{3^n}$.

Ratio Test:

$$\lim_{n \to \infty} \left| \frac{(-1)^{n+1}(n+1)!(x-5)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n!(x-5)^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{(n+1)\cancel{n!}(x-5)}{3\cancel{n!}} \right|$$

$$= \infty$$

Radius of convergence is 0.
Interval of convergence is $\{5\}$.

4. (4 POINTS) Write a series which is equivalent to $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ with the index of summation starting at 1.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \boxed{\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}}$$

Part II: (25 points) Problems 5-6. Find a geometric power series for the function, centered at c, and determine the interval of convergence.

5. (10 POINTS) $f(x)=\dfrac{4}{8-x}$, $c=2$

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r|<1$$

$$\frac{4}{8-x} = \frac{4}{8-2-(x-2)}$$

$$= \frac{4/6}{\frac{6}{6}-\frac{(x-2)}{6}}$$

$$= \frac{2/3}{1-\frac{x-2}{6}}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{2}{3}\left(\frac{x-2}{6}\right)^n, \quad -4<x<8}$$

I.O.C.

$$\left|\frac{x-2}{6}\right|<1$$

$$-1<\frac{x-2}{6}<1$$

$$-6<x-2<6$$

$$-4<x<8$$

6. (15 POINTS) $f(x)=\dfrac{4x}{x^2+2x-3},\ c=0$

$$\frac{4x}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1} = \frac{3}{x+3} + \frac{1}{x-1}$$

$$4x = A(x-1) + B(x+3)$$

$$4x = Ax - A + Bx + 3B$$

$$4x + 0 = (A+B)x + (-A+3B)$$

$$A + B = 4$$
$$-A + 3B = 0$$
$$4B = 4$$
$$B = 1$$

so $A = 3$

$$\frac{3/3}{\frac{3}{3}+\frac{x}{3}} = \frac{1}{1+\frac{x}{3}} = \frac{1}{1-\left(-\frac{x}{3}\right)} = \sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n, \quad -3<x<3$$

$$\frac{1}{x-1} = \frac{1/-1}{\frac{-1}{-1}+\frac{x}{-1}} = \frac{-1}{1-x} = -\frac{1}{1-x} = -\sum_{n=0}^{\infty} x^n, \quad -1<x<1$$

So, $$\frac{3}{3+x} + \frac{1}{x-1} = \sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n - \sum_{n=0}^{\infty} x^n$$

$$= \boxed{\sum_{n=0}^{\infty}\left[\left(-\frac{1}{3}\right)^n - 1\right]x^n, \quad -1<x<1}$$

Part III: (15 points) Problem 7. **Use the definition of Taylor series** to find the Taylor series for the function, centered at *c*. Be sure to find the interval of convergence and test the endpoints. You may not use a series known from a list. Hint: you may need to integrate or differentiate.

7. $f(x)=\dfrac{1}{1-x},\ c=2$

$f(x)=(1-x)^{-1}$ $\qquad f(2)=-1$

$f'(x)=-(1-x)^{-2}(-1)=(1-x)^{-2}$ $\qquad f'(2)=1$

$f''(x)=-2(1-x)^{-3}(-1)=2(1-x)^{-3}$ $\qquad f''(2)=-2$

$f'''(x)=-3\cdot2(1-x)^{-4}(-1)=3.2(1-x)^{-4}$ $\qquad f'''(2)=3\cdot2$

$f^{(4)}(x)=-4\cdot3\cdot2(1-x)^{-5}(-1)=4\cdot3\cdot2(1-x)^{-5}$ $\qquad f^{(4)}(2)=-4\cdot3\cdot2$

$$\frac{1}{1-x}=-1+1(x-2)-\frac{2(x-2)^2}{2!}+\frac{3\cdot2(x-2)^3}{3!}-\frac{4\cdot3\cdot2(x-2)^4}{4!}+-\cdots$$

$$\frac{1}{1-x}=-1+(x-2)-(x-2)^2+(x-2)^3-(x-2)^4+\cdots$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1+1}(x-2)^{n+1}}{(-1)^{n+1}(x-2)^n}\right|$$

$$=\lim_{n\to\infty}|x-2|<1\rightarrow 1<x<3$$

Test $x=1$:

$\sum_{n=0}^{\infty}(-1)^{n+1}(-1)^n=\sum_{n=0}^{\infty}(-1)^{2n+1}$ diverges by nth term test

Test $x=3$:

$\sum_{n=0}^{\infty}(-1)^{n+1}(1)^n=\sum_{n=0}^{\infty}(-1)^{n+1}$ diverges by nth term test

$$\boxed{\frac{1}{1-x}=\sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n,\quad 1<x<3}$$

Part IV: (30 points/15 points each) Problems 8-9. Solve the following problems as indicated. You do not need to find the interval of convergence.

8. Use the binomial series

$$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots$$

to find the Maclaurin series for the function $f(x) = \dfrac{1}{(1+x)^4}$.

$$f(x) = (1+x)^{-4}, \quad k = -4$$

$$(1+x)^{-4} = 1 - 4x + \frac{(-4)(-5)x^2}{2!} + \frac{(-4)(-5)(-6)x^3}{3!} + \frac{(-4)(-5)(-6)(-7)x^4}{4!} - \cdots$$

$$(1+x)^{-4} = 1 - 4x + \frac{5\cdot 4\, x^2}{2!} - \frac{6\cdot 5\cdot 4\, x^3}{3!} + \frac{7\cdot 6\cdot 5\cdot 4\, x^4}{4!} - \cdots$$

$$(1+x)^{-4} = 1 - 4x + \frac{(-1)^2\cdot 5!\, x^2}{3!\,2!} + \frac{(-1)^3\, 6!\, x^3}{3!\,3!} + \frac{(-1)^4\, 7!\, x^4}{3!\,4!} - \cdots$$

$$(1+x)^{-4} = \frac{(-1)^0\, 3!\, x^0}{3!\,0!} + \frac{(-1)^1\, 4!\, x^1}{3!\,1!} + \frac{(-1)^2\, 5!\, x^2}{3!\,2!} + \frac{(-1)^3\, 6!\, x^3}{3!\,3!} + \frac{(-1)^4\, 7!\, x^4}{3!\,4!} + \cdots$$

$$\boxed{(1+x)^{-4} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+3)!\, x^n}{3!\, n!}}$$

9. Use the series $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$ to find the Maclaurin series for the function $f(x) = x \sin x$.

$f(x) = x \sin x$

$f(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$f(x) = x \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right]$

$f(x) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}}$$