9.10: Taylor and Maclaurin Series

When you finish your homework you should be able to…

- π Find a Taylor series or a Maclaurin series for a function.
- π Find a binomial series.
- π Use a basic list of Taylor series to derive other power series.

WARM-UP: Find the 8th degree Maclaurin polynomial for the function $f(x) = \cos x$. $= f^{(4)}(x)$

$f'(x) = -\sin x = f^{(5)}(x)$

$f''(x) = -\cos x = f^{(6)}(x)$

$f'''(x) = \sin x = f^{(7)}(x)$

$f^{(8)}(x) = \cos x$

$f(0) = f^{(4)}(0) = f^{(8)}(0) = 1$

$f'(0) = f^{(5)}(0) = 0$

$f''(0) = f^{(6)}(0) = -1$

$f'''(0) = f^{(7)}(0) = 0$

$$P_8(x) = 1 + 0x - \frac{1x^2}{2!} + \frac{0x^3}{3!} + \frac{1x^4}{4!} + \frac{0x^5}{5!} - \frac{1x^6}{6!} + \frac{0x^7}{7!} + \frac{1x^8}{8!}$$

$$P_8(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

Now let's see if we can form a power series!

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad (-\infty, \infty)$$

What about that interval of convergence?

Ratio test:

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{x^2(2n)!}{(2n+2)(2n+1)(2n)!}\right| = 0 < 1$$

IOC $(-\infty, \infty)$

Theorem: The Form of a Convergent Power Series

If f is represented by a power series $f(x) = \sum a_n (x-c)^n$ for all x in an open interval I containing c, then

and

Definition of Taylor and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

is called the ________________ series for ______ at ____. If _________,

then the series is the ___________________ series for _____.

Example 1: Find the Taylor series, centered at c, for the function.

a. $f(x) = e^{-4x},\ c = 0$

b. $f(x)=\dfrac{1}{1-x},\ c=2$

Theorem: Convergence of Taylor Series

If $\lim_{n\to\infty} R_n = 0$ for all x in the interval I, then the Taylor series for f converges and equals $f(x)$.

Example 2: Prove that the Maclaurin series for $f(x) = \cos x$ converges to $f(x)$ for all x.

Binomial Series

Let's check out the function $f(x)=(1+x)^k$, where k is a rational number. What do you think the Maclaurin series is for this function? Guess what…YOU KNOW HOW TO FIND IT!!! So, on your mark, get set, GO!

1. __________________ $f(x)$ a bunch of times and evaluate each

   __________________ at ______. Evil plan: ____________________ a

   ____________________.

2. Determine the ________________ of __________________...Don't forget to test the __________________!

3. Determine whether the series ___________________ to $f(x)$ within the ______________ of ________________.

Guidelines for Finding a Power Series

1. ____________________ $f(x)$ and _________________ each ____________________ at _____ until you find a ______________________.
2. Form the _______________ coefficient ____________________, and determine the ________________ of convergence for the ______________ series.
3. Determine whether the series __________________ to _________ within the interval of convergence.

Example 3: Find the Maclaurin series for the function using the binomial series.

a. $f(x) = \dfrac{1}{(1+x)^4}$

b. $f\left(x\right)=\sqrt{1+x^3}$

A Basic List of Power Series for Elementary Functions

FUNCTION	INTERVAL OF CONVERGENCE
$\frac{1}{x} =$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x =$	$0 < x \le 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x =$	$-1 \le x \le 1$
$\arcsin x =$	$-1 \le x \le 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$	$-1 < x < 1$*

*convergence at endpoints depends on k

Example 4: Find the Maclaurin series for the function using the basic list of power series for elementary functions.

a. $f(x) = \ln(1 + x^2)$

b. $f(x) = e^x + e^{-x}$

c. $f(x) = \cos^2 x$

d. $f(x) = x\cos x$

e. $f\left(x\right)=\cot x$

Example 5: Find the first four nonzero terms of the Maclaurin series for the function $f(x) = e^x \ln(1+x)$.

Example 6: Use a power series to approximate the value of the integral with an error less than 0.0001.

$$\int_0^{1/2} \arctan x^2 dx$$