When you finish your homework you should be able to...

- Determine infinite limits from the left and from the right
- Find and sketch the vertical asymptotes of the graph of a function

Warm-up: Evaluate the following limits analytically

1. $\lim_{x \to 0} \frac{\left[1/(x+4) - 1/4 \right]}{x}$

2. $\lim_{t \to 0} \frac{\sin 3t}{2t}$

3. $\lim_{\theta \to 0} \frac{\cos\theta \tan\theta}{\theta}$

DEFINITION OF INFINITE LIMITS

Let *f* be a function that is defined at every real number in some open interval containing *c* (except possibly at *c*). The statement

$$\lim_{x\to c} f(x)=\infty$$

means that for each $M>0$ there exists a $\delta>0$, such that $f(x)>M$ whenever $0<|x-c|<\delta$. Similarly, the statement

$$\lim_{x\to c} f(x)=-\infty$$

means that for each $N<0$ there exists a $\delta>0$, such that $f(x)<N$ whenever $0<|x-c|<\delta$. To define the **infinite limit from the left**, replace $0<|x-c|<\delta$ by $c-\delta<x<c$. To define the **infinite limit from the right**, replace $0<|x-c|<\delta$ by $c<x<c+\delta$.

Example 1: Determine the infinite limit.

1. $\displaystyle\lim_{x\to -3^-}\frac{x}{x^2-9}$

2. $\displaystyle\lim_{x\to 3^+}\sec\frac{\pi x}{6}$

DEFINITION OF VERTICAL ASYMPTOTE

If $f(x)$ approaches infinity or negative infinity as x approaches c from the right or the left, then the line $x=c$ is a **vertical asymptote** of the graph of f.

THEOREM: VERTICAL ASYMPTOTES

Let f and g be continuous on an open interval containing c. If $f(c)\neq 0$, $g(c)=0$, and there exists an open interval containing c such that $g(x)\neq 0$ for all $x\neq c$ in the interval, then the graph of the function given by

$$h(x)=\frac{f(x)}{g(x)}$$

has a **vertical asymptote** at $x=c$.

Example 2: Find the vertical asymptotes (if any) of the graph of the function.

1. $$g(\theta)=\frac{\tan\theta}{\theta}$$

2. $$h(x)=\frac{x^2-4}{x^3+2x^2+x+2}$$

THEOREM: PROPERTIES OF INFINITE LIMITS

Let b and L be real numbers and let f and g be functions such that

$$\lim_{x\to c} f(x)=\infty \qquad \text{and} \qquad \lim_{x\to c} g(x)=L$$

1. Sum or difference: $$\lim_{x\to c}\left[f(x)\pm g(x)\right]=\infty$$
2. Product: $$\lim_{x\to c}\left[f(x)g(x)\right]=\infty, \quad L>0$$
3. Quotient: $$\lim_{x\to c}\left[\frac{g(x)}{f(x)}\right]=0$$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$.

Example 3: Let $\lim_{x\to c} f(x)=-\frac{2}{3}$, $\lim_{x\to c} g(x)=\infty$ and $\lim_{x\to c} h(x)=5$. Determine the following limits:

1. $$\lim_{x\to c}\frac{f(x)}{g(x)}$$
2. $$\lim_{x\to c}\left[f(x)-g(x)\right]^2$$
3. $$\lim_{x\to c}\frac{\left[h(x)f(x)\right]}{3}$$